Non-commutative integration

on

locally compact quantum groups

Fourier theory - Gelfand pairs - Non-commutative L^p-spaces

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Contents

Preface

This thesis studies areas within harmonic analysis, quantum groups and non-commutative L^p-spaces. In particular, we study how the objects within the various areas interact and see how notions from one part enhance the study of another part. For this we use the techniques from operator algebras, in particular von Neumann algebras.

We shortly describe these different parts of mathematics below. This gives the reader some background to the material in this thesis. Afterwards we summarize the new results.

First of all, we study quantum groups. Quantum groups have their first roots in the 1950's when people looked at Hopf algebras. From this point the development took many different directions. Hence the word 'quantum group' means different things to different people nowadays. Fortunately, there is still an intersection of examples that lies within the common interest of all these people. The quantum groups we study are normally called operator algebraic quantum groups. Key point in their development was the discovery of $SU_q(2)$ (quantum $SU(2)$) and the general definition of a compact C*-algebraic quantum group by Woronowicz in the 1980's.

In this spirit, a quantum group is considered as a deformation of the algebra of continuous functions on a locally compact group. Basically this deformation is carried out as follows. Consider the continuous functions on a compact group G and call this $C(G)$. Using the Gelfand-Naimark theorem, the topology of G can be recovered from the C*-algebra $C(G)$. By equipping $C(G)$ with some extra structure, the group multiplication, unit and inverse can also be captured within $C(G)$. Basically, this is done by pulling back the respective maps on the group to the functions on the group.

The quantum part of the story is given by the following procedure. The algebra $C(G)$ can normally be given in terms of generators subject to a finite set of relations. In that way $C(G)$ can be seen as (a certain closure of) a universal algebra. In many cases, one can deform the relations that characterize this algebra. It depends on the group if a 'suitable' deformation is available. The object one ends up with is a (generally non-commutative) C*-algebra with additional structure resembling $C(G)$, see Definition 1.0.1. This is what we call a quantum group.

Around 2000 Kustermans and Vaes settled a suitable framework for the theory of *locally* compact quantum groups. The framework includes a C*-algebraic and von Neumann algebraic approach.

One of the main motivations for this direction in the development of quantum groups is the intense relation between quantum groups and harmonic analysis. Let us mention two important results that have a strong connection with this thesis.

First of all, the definition of a quantum group as given by Kustermans and Vaes is motivated by the classical Pontrjagin duality theorem. The theorem states the following. Let G be a locally compact group. Let $\hat{G}$ denote the equivalence classes of all continuous irreducible unitary representations of G. Then $\hat{G}$ can be equipped with a group structure by means of pointwise multiplication. $\hat{G}$ is also called the Pontrjagin dual group. Pontrjagin duality states that the double Pontrjagin dual of G is isomorphic to G itself. The Kustermans-Vaes defintion of a locally compact quantum group features a far reaching generalization of this result, which in particular gives Pontrjagin duality for any locally compact group.

Secondly, a very classical result is given by the Peter-Weyl theorem. It states that for a compact group G, the left regular representation λ decomposes as a direct sum

$$\lambda = \bigoplus_{\pi \in \hat{G}} \dim(\pi) \cdot \pi, \tag{1}$$

i.e. every irreducible unitary representation π appears as many times in the decomposition of λ as the dimension of π.

The Peter-Weyl theorem has many different generalizations. For a special class of *locally* compact groups that are of type I (see [22] for the definition), there exists a similar decomposition as (1). However, the direct sum changes for its continuous counterpart: a direct integral. This decomposition is called the Plancherel decomposition or Plancherel theorem and plays an important role in Sections 2 - 4 of this thesis.

In this thesis, the reader finds many other examples of relations between quantum groups and harmonic analysis. Our particular focus is the relationship with Fourier theory and theorems dealing with decompositions of representations into irreducible ones.

Finally, we use the theory of non-commutative integration. We replace measure spaces with integrals by weights on von Neumann algebras.

Von Neumann algebras are special subalgebras of the bounded operators on a Hilbert space. They were first studied by Murray and von Neumann around 1940 in order to understand quantum physics. A weight φ on a von Neumann algebra M is a certain unbounded functional on the positive part of M, i.e. $\varphi : M^+ \mapsto [0, \infty]$. φ is called a trace when,

$$\varphi(A^*A) = \varphi(AA^*), \qquad A \in M. \tag{2}$$

In this thesis we are particularly interested in the weights that are *not* a trace. Working with these weights requires a tool that was developed in the 1960's and is now known as Tomita-Takesaki theory. This theory gave an immense new impulse to the study of von Neumann algebras. In particular, the non-commutative L^p-spaces that we study in Chapters 5 and 6 are one of the offsprings of Tomita-Takesaki theory. They were defined (amongst others) by Haagerup as well as Connes and Hilsum.

For quantum groups, the Haar measure on a group translates into a Haar weight on a von Neumann algebraic quantum group. The Haar weights in this thesis are typically not traces and it is for this reason that the non-tracial theory is essential.

Let us give an overview of the new results obtained in this thesis.

Chapter 2. For a general weight φ on a von Neumann algebra M, one can say to what extent the equation (2) is violated. That is, there exists a one-parameter group of automorphisms $\mathbb{R} \to \mathrm{Aut}(M) : t \mapsto \sigma_t^\varphi$ together with a class of operators $A \in M$ for which the map

$$\mathbb{R} \to M : t \mapsto \sigma_t^\varphi(A),$$

can be extended analytically to $\mathbb{C}$ and for which moreover a twisted relation holds:

$$\varphi(A^*A) = \varphi(A\sigma_{-i}(A^*)).$$

Obviously, we avoid the technicalities here. They can be found in Appendix A.2. The one-parameter group $t \mapsto \sigma_t^\varphi$ is called the modular automorphism group of the weight φ. This twisted relation is a consequence of Tomita-Takesaki theory and is an imperative tool for handling weights that are not tracial. In case φ is a trace, σ_t^φ is the identity automorphism for all $t \in \mathbb{R}$.

Chapter 2 describes the modular automorphism group of a Haar weight of a quantum group in terms of the Plancherel decomposition of the quantum group. Recall that the Plancherel decomposition gives a direct integral decomposition of the left regular representation. The intertwiner that decomposes the left regular representation is defined in terms of a field of positive self-adjoint operators, so called Duflo-Moore operators. It turns out that the modular automorphism group σ_t^φ can be expressed in terms of these Duflo-Moore operators; we find an explicit formula. This phenomenon has no (non-trivial) analogy in the classical group setting.

We also give some applications at the end of Section 2. In particular we give a new method to find the Haar weight of the dual of a quantum group that satisfies the assumptions of the quantum Plancherel theorem. We make this explicit for the operator algebraic deformation of $SU(1,1)$.

The results of this chapter are published in a paper with Erik Koelink [10], except for the final part of Section 2.4.

Chapters 3 and 4. Consider a locally compact group G with compact subgroup K. The convolution product on $L^1(G)$ restricts to $L^1(K\backslash G/K)$, i.e. the L^1-functions on G that are invariant under the actions of K by left and right translation. When the convolution algebra $L^1(K\backslash G/K)$ is commutative, the pair (G, K) is called a Gelfand pair.

Gelfand pairs form an important part of harmonic analysis. A cornerstone of the theory is a theorem that states that a special class of functions $f \in L^1(K\backslash G/K)$ can be decomposed as an integral of elementary matrix coefficients. That is, there exists a decomposition

$$f(x) = \int_\Omega c(\pi)\langle \pi(x)\xi_\pi, \xi_\pi\rangle d\mu(\pi),$$

where μ is a measure on Ω, the space of irreducible representations π of G that admit a (necessarily unique up to scalar multiplication) unit vector ξ_π that is fixed for the restriction of π to K. Here $c(\pi)$ is the constant given by $\int_G f(g)\pi(g)dg$. This decomposition is also called the Plancherel-Godement theorem.

Chapter 3 gives the necessary conceptual framework for locally compact quantum Gelfand pairs. The chapter culminates in a quantum analogue of the Plancherel-Godement theorem. Such a theorem was already known for the group case (as described above), as well as for the compact quantum group case, see:

	compact	non-compact
groups	$\times$	$\times$
quantum groups	$\times$	

For the bottom right corner a few examples are available on an algebraic level. These all precede the work of Kustermans and Vaes and therefore a general theory was unavailable. We construct this theory in Chapter 3.

In section Chapter 4 we give a new example of a quantum Gelfand pair and its spherical Fourier transform. Namely, we treat the example of $SU_q(1,1)_{\text{ext}}$ with the circle as its subgroup. What is particularly interesting about this example, is that it is not a Gelfand pair in the traditional sense. However, it is very close to being a Gelfand pair as we explain. We show that by means of gradings we can obtain exactly the same type of results as for classical Gelfand pairs.

The results of these chapters have been published in [8].

Chapter 5. Consider the following remarkable fact. For $f \in L^1(\mathbb{R})\cap L^p(\mathbb{R})$ with $1 \leq p \leq 2$,

$$\hat{f}(y) = \int_\mathbb{R} f(z)e^{-izy}dz,$$

is an element of $L^q(\mathbb{R})$, with $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the mapping $L^p(\mathbb{R}) \to L^q(\mathbb{R}) : f \mapsto \hat{f}$ is norm decreasing.

Chapter 5 considers a far reaching generalization of this result. We consider the non-commutative L^p-spaces associated with the von Neumann algebra of a quantum group, say $L^p(M)$, where $1 \leq p \leq \infty$. For the precise definition of $L^p(M)$ we refer to Chapter 5. However, let us emphasize that in the naive sense

$$L^p(M) \cap L^{p'}(M) = \{0\} \qquad \text{if } p \neq p'.$$

Using the techniques of interpolation spaces, we are able to embed $L^p(M)$ for different p in a single Banach space E. This gives a non-trivial meaning to the intersections of $L^p(M)$ for different p by viewing them as subspaces of E. In Chapter 5 we determine such intersections. Using those results, we obtain the main theorem of Chapter 5: we construct a Fourier transform on $L^p(M)$ for $1 \leq p \leq 2$. We also introduce a convolution product in the L^p-setting and show that the Fourier transform transfers the convolution into a product. Next, we consider pairings between the L^p-space of a quantum group and the L^p-space of its dual. Finally, we show how the L^p-Fourier transform distinguishes the interpolation structure. That is, the real part of the interpolation parameter introduced by Kosaki [54] and Izumi [40] is determined by the L^1-Fourier transform.

Most of the results of this chapter are contained in [7], which will be published shortly. However, the following parts are new. Theorem 5.6.9 was stated in [7] but the technical part of its proof was omitted. We present this part here. Section 5.8 is new and resolves a certain asymmetry that is hidden in [7].

Chapter 6. We study L^p-spaces associated with hyperfinite factors of type III. These are separable Banach spaces and therefore it is a natural question wether or not we can construct a Schauder basis in those spaces. Recall that a Schauder basis in a Banach space E is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that for every $x \in E$ we have a unique decomposition

$$x = \sum_{n=0}^{\infty} \alpha_n x_n, \qquad \alpha_n \in \mathbb{C}.$$

We use recent results by Haagerup, Junge and Xu [38], [41] to find a particular Schauder basis in non-commutative L^p-spaces. That is, we find a Walsh basis in the non-commutative L^p-spaces of hyperfinite III_λ factors, where $1 < p < \infty$ and $0 < \lambda \leq 1$.

The results are contained in the forthcoming paper with Denis Potapov and Fedor Sukochev [11].

Structure and logical dependence

The core of this thesis consists of three parts. An introduction, the new results and the appendix. The introduction should give the reader sufficient background on quantum groups to understand the results and proofs in this thesis. The appendix summarizes the main technical ingredients. This concerns mostly the theory of weights on von Neumann algebras, i.e. Tomita-Takesaki theory, Radon-Nikodym derivatives, et cetera (see [75]).

The results in Chapters 2 to 6 are new. Most of these results are also contained in a forthcoming paper as indicated above. We have changed the (chronological) order in which these papers appeared for a thematic order. Historically, the order of the chapters would be 2 – 5 – 3 – 4 – 6.

Each chapter is reasonably self-contained, except for Chapter 4, which works out an example of Chapter 3. Furthermore, the reader who is unfamiliar with non-commutative L^p-spaces and interpolation spaces might find it worthwhile to read the first sections of Chapter 5 for a better understanding of Chapter 6. However, this is not strictly necessary since proper references are given.

Preliminaries

This thesis is intended for readers with a comfortable knowledge in operator algebras. The books by Pedersen [67] and Murphy [64] are good references for many of the basic results we use. In addition, the still quite up-to-date book by Stratila and Zsido [73] gives more than enough introduction. For the theory of weights on von Neumann algebras, our main reference is Takesaki's book [75]. We give proper references when we need results from this source. To keep the thesis more self-contained, we summarize some of the main constructions such as spatial derivatives, cocycle derivatives, et cetera, in the appendix. For direct integration we refer to Dixmier's book on von Neumann algebras [21], see also the appendix.

For the theory of quantum groups we would recommend Timmermann's book [79]. Other good sources are the lecture notes by Kustermans [57] or the different approach by Van Daele [93]. And, of course, one may also look at Kustermans' and Vaes' original papers [59], [60]. In the introduction, we outline their main concepts and we state the more specific lemmas needed. We do not comment on the relations with Hopf algebras and other branches of quantum groups, for which we refer again to [79].

We do not assume that the reader is familiar with non-commutative L^p-spaces. A brief introduction is contained in Chapter 5. However, for a more deliberate introduction we find [77] most useful.

Notational conventions

Throughout the thesis, we use the following general notation.

General notation. $\mathbb{T}$ denotes the complex unit circle. $\mathbb{N}$ denotes the natural numbers starting with 0. The character Σ always denotes some kind of flip map. We state its domain when we use it. For a C*-algebra A, we use $M(A)$ for its multiplier algebra.

Von Neumann algebras and weights. M will always denote a von Neumann algebra. M_* and M^+ denote its predual and positive cone. If φ is a weight on M with modular automorphism group σ, we use the notations:

$$\begin{aligned}
\mathfrak{n}_\varphi &= \{x \in M \mid \varphi(x^*x) < \infty\}, \\
\mathfrak{m}_\varphi &= \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi. \\
\mathcal{T}_\varphi &= \{a \in M \mid a \text{ is analytic for } \sigma \text{ and } \sigma_z(a) \in \mathfrak{n}_\varphi^* \cap \mathfrak{n}_\varphi, z \in \mathbb{C}\}.
\end{aligned}$$

We freely use the fact that $\mathfrak{n}_\varphi$ is a left ideal. For $a, b \in \mathfrak{n}_\varphi$ we use $a\varphi b^*$ to denote the normal functional on M given by $(a\varphi b^*)(x) = \varphi(b^* x a), x \in M$. In fact, we recall this notation several times.

Hilbert spaces. Hilbert spaces are denoted by calligraphic symbols. Every inner product is linear in the first entry and anti-linear in the second. If $\mathcal{H}$ is a Hilbert space, $\xi, \eta \in \mathcal{H}$ and x is an element in a von Neumann algebra M acting on $\mathcal{H}$, then we use the notation $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$. We also use $\theta_{\xi,\eta}$ for the rank one operator on $\mathcal{H}$ defined by $\theta_{\xi,\eta} v = \langle v, \eta\rangle \xi, v \in \mathcal{H}$. We use $^\perp$ to denote a complement in a Hilbert space.

Closures. For two unbounded operators x and y on a Hilbert space, we denote $x \cdot y$ for the closure of the product, which exists in the case we use this notation. We also use $[x]$ to denote the closure of x, in case x is preclosed.

Homomorphisms and representations. The symbol ι is used for the identity map. It should always be clear from the context what the domain of ι is. We omit the symbol $\circ$ to denote the composition of maps. This means that we just write πT for the composition of, for example a representation π and a conditional expectation value T. Whenever we speak about a homomorphism or representation of a Banach-$*$-algebra or a C*-algebra, this means $*$-homomorphism, unless explicitly stated otherwise. Representations are always assumed to represent on a Hilbert space. Similarly, all (co)representations of (quantum) groups are asummed to be unitary, unless explicitly stated otherwise. For a C*-algebra A, we use the notation $\mathrm{IR}(A)$ to denote the equivalence classes of irreducible representations of A.

Chapter 1

Quantum groups

A quantum group should be considered as a non-commutative analogue of a group or rather a deformation of the function algebra on a group. Let us sketch the idea of a stereotypical example. Consider the C*-algebra $C(G)$ of continuous complex valued functions on a compact Lie group G. By the celebrated Gelfand-Naimark theorem, we know that all the topological data of our Lie group is contained in this C*-algebra. With some extra effort, one can equip this algebra with an additional structure such that also the group structure can be recovered.

The *quantum* part of the story is that one is able to *deform* this algebra. In many cases, the algebra $C(G)$ can be given in terms of a finite set of generators, satisfying certain relations. Then, one deforms the relations that characterize the Lie group by means of a parameter commonly denoted by 'q' (it depends on the Lie group if a 'reasonable' deformation exists). In that way examples refered to as 'quantum SU(2)' or simply $SU_q(2)$ arize. These objects are not functions on a group anymore but merely (non)-commutative algebras with some additional structure. One should think of it as a non-commutative space or a non-commutative Lie group, or (probably) in the words of Drinfel'd, a *quantum group*.

It turns out that an incredible amount of the structure of, for example, $SU(2)$ is preserved under the deformation. These include *geometric* properties; one can define Dirac operators on such a deformation [16], [6]. As such quantum groups form one of the most important examples for non-commutative geometers. On the other hand, many tools from abstract harmonic analysis have a suitable interpretation in the deformed setting. These are only two examples; quantum groups appear at more places. And, of course, these motivations are not separable worlds, but strongly interact. The reader will find numerous other examples in the present thesis.

During its development, quantum groups appear in many different guises. It is good to mention that 'quantum group' means different things to different people nowadays. The quantum groups we consider are normally refered to as *operator algebraic quantum groups* or *locally compact quantum groups*. Other approaches appear from for example a purely algebraic point of view or from a categorical point of view. The intersection is probably a class of examples that is 'interesting' to any of these branches. The root of all these branches can be found 1950's, when Hopf algebraic structures were introduced. The word 'quantum group' was introduced much later, in the 1980's allegedly by Drinfel'd. We summarize some historical key points in the development of operator algebraic quantum groups.

First of all, an important step was made in the 1970's with the development of Kac algebras. The main motivation was to generalize the celebrated Pontrjagin duality theorem: for an abelian locally compact group G, the set of irreducible representations forms a group again under pointwise multiplication. This group is refered to as the dual group. Pontrjagin duality states that the double dual of a group is isomorphic to the group itself. For non-abelian groups these constructions do not make sense, since one cannot (a priori) multiply representations of different dimension. Therefore, people started looking for a larger category than groups in which a more general notion of a dual object can be defined. The structure was found independently due to Kac and Vainerman [90], [91] and on the other hand by Enock and Schwarz, see [25]. The structure is what is now called a Kac algebra.

The next main step is the discovery of $SU_q(2)$ by Woronowicz [98]. $SU_q(2)$ is not a Kac algebra, but still carries all the quantum group properties one could expect. This shows that the category of Kac algebras may be to small and should be replaced. This lead to the operator algebraic definition by Woronowicz of what is now known as a compact quantum group, see [96], [97] and [98]. The definition gave a C*-algebraic interpretation to deformations of compact Lie groups. For completeness, we state it here.

Definition 1.0.1. A *compact quantum group* in the sense of Woronowicz is a pair (A, Δ) of a unital C*-algebra A and a unital $*$-homomorphism $\Delta : A \to A \otimes A$ (minimal tensor product) called the *comultiplication*, that satisfies *coassociativity*:

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta.$$

Moreover, the cancellation laws hold:

$$\overline{\mathrm{span}}\Delta(A)(A \otimes 1_A) = \overline{\mathrm{span}}\Delta(A)(1_A \otimes A) = A \otimes A.$$

At that moment a suitable interpretation of a *locally* compact quantum group was absent. One way to go was to give a definition directly in terms of the left regular representation, the so called multiplicative unitary. Important steps in this direction have been made, amongst others, by Baaj and Skandalis [2].

Around 2000 Kustermans and Vaes gave a satisfactory definition in the locally compact setting. The definition is satisfactory, in the sense that:

- It is short and not too technical;

- It generalizes the celebrated study of compact quantum groups by Woronowicz; furthermore, it includes some special examples which were thought of as being quantum groups;

- It allows a generalization of the Pontrjagin duality theorem.

We emphasize that the Kustermans-Vaes approach does not at all make the earlier studies of quantum groups redundant. In fact, the underlying Hopf algebraic interpretation plays an important role in many of the examples. Furthermore, most things which are known now for compact quantum groups can perfectly be understood in the sense of Woronowicz.

However, it is the Kustermans-Vaes definition of a locally compact quantum group, that will have a strong focus in the present thesis.

1.1 Von Neumann algebraic quantum groups

This section recalls the von Neumann algebraic definition of a locally compact quantum group as was given by Kustermans and Vaes [59], [60]. For good introductions to the theory of locally compact quantum groups, we refer the reader to [57], [79] or [93].

Definition 1.1.1. A *locally compact quantum group* (M, Δ) consists of the following data:

1. A von Neumann algebra M;

2. A unital, normal $*$-homomorphism $\Delta : M \to M \otimes M$ called the *coproduct* satisfying the *coassociativity* relation $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;

3. Two normal, semi-finite, faithful weights φ, ψ on M so that φ is *left invariant* and ψ is *right invariant*, i.e.

$$\begin{aligned} \varphi\left((\omega \otimes \iota)\Delta(x)\right) &= \varphi(x)\omega(1), \qquad \omega \in M_*^+,\ x \in \mathfrak{m}_\varphi^+, \\ \psi\left((\iota \otimes \omega)\Delta(x)\right) &= \psi(x)\omega(1), \qquad \omega \in M_*^+,\ x \in \mathfrak{m}_\psi^+ \qquad . \end{aligned}$$

   φ is called the *left Haar weight* and ψ the *right Haar weight*.

Note that we suppress the Haar weights in the notation. In fact, if the Haar weights exist, then they are unique up to multiplication by a positive scalar.

A locally compact quantum group is called *unimodular* if the left Haar weight equals the right Haar weight up to a scalar. We call a quantum group *compact* if the Haar weights are states. Compact quantum groups are always unimodular. Moreover, if a locally compact quantum group is compact, the underlying reduced C^*-algebraic quantum group introduced in Section 1.4 is a compact quantum group in the sense of Woronowicz.

Example 1.1.2. In order to reflect to the classical situation of a locally compact group, we include the following example. Let G be a locally compact group. Consider $M = L^\infty(G)$ and define the coproduct $\Delta_G : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \simeq L^\infty(G \times G)$ by putting

$$(\Delta_G(f))(x,y) = f(xy).$$

φ and ψ are given by integrating against the left and right Haar measures respectively. In this way $(L^\infty(G), \Delta_G)$ is a locally compact quantum group.

We let $(\mathcal{H}, \pi, \Lambda)$ denote the GNS-construction with respect to the left Haar weight φ. We may assume that M acts on the GNS-space $\mathcal{H}$ and therefore we omit the map π in the notation. We use ∇ to denote the modular operator and J to denote the modular conjugation associated with φ. σ denotes the modular automorphism group of φ and σ^ψ denotes the modular automorphism group of ψ. We use the notation $(\mathcal{H}, \pi_\psi, \Gamma)$ for the GNS-representation of ψ. Recall that the Hilbert space $\mathcal{H}$ can be taken the same as the GNS-space for φ.

Theorem-Definition 1.1.3. *There exists a unique unitary operator $W \in B(\mathcal{H} \otimes \mathcal{H})$ defined by:*

$$W^* \left(\Lambda(a) \otimes \Lambda(b)\right) = \left(\Lambda \otimes \Lambda\right) \left(\Delta(b)(a \otimes 1)\right), \qquad a, b \in \mathfrak{n}_\varphi.$$

W is known as the multiplicative unitary.

The multiplicative unitary satisfies the *pentagon equation*

$$W_{12} W_{13} W_{23} = W_{23} W_{12}$$

in $B(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$. Furthermore, it implements the comultiplication in the following way:

$$\Delta(x) = W^*(1 \otimes x) W, \qquad x \in M. \tag{1.1}$$

Also, the von Neumann algebra M can be recovered from W by

$$M = \overline{\{(\iota \otimes \omega)(W) \mid \omega \in B(\mathcal{H})_*\}}^{\sigma-\text{strong}-*}.$$

In particular $W \in M \otimes B(\mathcal{H})$. Since the Haar weights on (M, Δ) are unique up to positive scalar multiplication, W fully captures the structure of our quantum group.

Theorem-Definition 1.1.4. *To (M, Δ) one can associate an unbounded map called the antipode $S : Dom(S) \subseteq M \to M$. It can be defined as the σ-strong-$*$ closure of the map:*

$$(\iota \otimes \omega)(W) \mapsto (\iota \otimes \omega)(W^*), \tag{1.2}$$

with $\omega \in B(\mathcal{H})_*$.

We mention that in [59] and [60] other equivalent definitions are given, but for our purposes the present one will suffice. By Lemma A.6.1 we see that S is also σ-weakly closed.

Since the antipode is usually unbounded, it is useful to consider its polar decomposition. That is, one can prove that there exists a unique $*$-anti-automorphism $R : M \to M$ and a unique strongly continuous one-parameter group of $*$-automorphisms $\tau : \mathbb{R} \to \mathrm{Aut}(M)$ and a constant $\nu \in \mathbb{R}^+$ such that

$$S = R\tau_{-i/2}, \quad R^2 = \iota, \quad \tau_t R = R\tau_t, \quad \varphi\tau_t = \nu^{-t}\varphi, \qquad t \in \mathbb{R}.$$

R is called the *unitary antipode* and τ is called the *scaling group*. The constant ν is called the *scaling constant*. Moreover,

$$\Delta R = \Sigma_{M,M}(R \otimes R)\Delta, \quad \psi = \varphi R, \tag{1.3}$$

where $\Sigma_{M,M} : M \otimes M \to M \otimes M$ is the flip.

Using the relative invariance property of the left Haar weight with respect to the scaling group, we define P to be the positive operator on $\mathcal{H}$ such that

$$P^{it}\Lambda(x) = \nu^{\frac{t}{2}}\Lambda(\tau_t(x)), \qquad t \in \mathbb{R}, x \in \mathfrak{n}_\varphi.$$

Also, one can prove that there exists a unique operator $\delta > 0$, called the *modular element*, which is affiliated with M such that $\psi = \varphi_\delta$, see Appendix A.4. So formally $\psi(x) = \varphi(\delta^{1/2}x\delta^{1/2})$. Moreover, $\sigma_t(\delta) = \nu^t\delta$.

We use the notation

$$M_*^\sharp = \{\omega \in M_* \mid \text{There exists } \theta \in M_* \text{ s.t. } (\theta \otimes \iota)(W) = (\omega \otimes \iota)(W)^*\}. \tag{1.4}$$

For $\omega \in M_*^\sharp$, ω^* is defined by $(\omega^* \otimes \iota)(W) = (\omega \otimes \iota)(W)^*$. In that case $\omega^*(x) = \overline{\omega}(S(x)), x \in \mathrm{Dom}(S)$. For $\omega \in M_*^\sharp$, we set $\|\omega\|_* = \max\{\|\omega\|, \|\omega^*\|\}$. $M_*^\sharp$ becomes a Banach-$*$-algebra with this norm. By smearing elements with respect to the scaling group, one can proof the following lemma.

Lemma 1.1.5 (Lemma 2.5 of [60]). *The set $M_*^\sharp$ is dense in M_*.*

1.2 Pontrjagin duality

One of Kustermans and Vaes their motivations for their definition of a locally compact quantum group was to extend the Pontrjagin duality theorem for locally compact abelian groups to the far more general setting of quantum groups. They proved that for every locally compact quantum group (M, Δ) there exists a dual locally compact quantum group $(\hat{M}, \hat{\Delta})$, so that $(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$.

By construction,

$$\hat{M} = \overline{\{(\omega \otimes \iota)(W) \mid \omega \in B(\mathcal{H})_*\}}^{\sigma-\mathrm{strong}-*}.$$

This implies that $W \in M \otimes \hat{M}$. Furthermore, $\hat{W} = \Sigma W^* \Sigma$, where Σ denotes the flip on $\mathcal{H} \otimes \mathcal{H}$. The dual coproduct can be given by the dualized formula $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}, x \in \hat{M}$. For $\omega \in M_*$, we use the standard notation

$$\lambda(\omega) = (\omega \otimes \iota)(W).$$

Let us describe the construction of the dual Haar weights. We let

$$\mathcal{I} = \{\omega \in M_* \mid \Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi, \text{ is bounded } \}$$

By the Riesz theorem, for every $\omega \in \mathcal{I}$, there is a unique vector denoted by $\xi(\omega) \in \mathcal{H}$ such that

$$\omega(x^*) = \langle \Lambda(x), \xi(\omega) \rangle, \qquad x \in \mathfrak{n}_\varphi. \tag{1.5}$$

The dual left Haar weight $\hat{\varphi}$ is defined to be the unique normal, semi-finite, faithful weight on $\hat{M}$, with GNS-construction $(\mathcal{H}, \iota, \hat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a σ-strong-$*$/norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega), \omega \in \mathcal{I}$. We let $\hat{\nabla}$ and $\hat{J}$ denote the modular operator and modular conjugation associated with $\hat{\varphi}$.

Analogously, we can define a set

$$\mathcal{I}_R = \{\omega \in M_* \mid \Gamma(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\psi, \text{ is bounded } \},$$

and we define for $\omega \in \mathcal{I}_R$ the vector $\xi_R(\omega)$ by the property $\langle \xi_R(\omega), \Gamma(x) \rangle = \omega(x^*)$, where $x \in \mathfrak{n}_\psi$.

Note that we may introduce the dual antipode $\hat{S}$ by means of the dualized formula (1.2) and similarly, we let $\hat{R}$ and $\hat{\tau}$ be the dual unitary antipode and dual scaling group. Then, the dual right Haar weight is given by $\hat{\psi} = \hat{\varphi}R$. Any other object in this thesis associated with the dual quantum group will be equipped with a hat.

We need the following lemma several times.

Lemma 1.2.1 (Lemma 8.5 of [59]). *Let $a, b \in \mathcal{T}_\varphi$. Then, $a\varphi b \in \mathcal{I}$ and $\xi(a\varphi b) = \Lambda(b\sigma_{-i}(a))$. In particular, the vectors $\xi(\omega), \omega \in \mathcal{I}$, are dense in $\mathcal{H}$.*

Moreover, we record the following relations.

Lemma 1.2.2 (Proposition 2.1 and Corollary 2.2 of [60]). *We have the following relations:*

$$(\tau_t \otimes \hat{\tau}_t)(W) = W, \qquad (R \otimes \hat{R})(W) = W^*.$$

Lemma 1.2.3 (Lemma 8.8 and Proposition 8.9 of [59]). *We have $\hat{P} = P$ and furthermore, it satisfies the relations:*

$$\hat{\nabla}^{it} = P^{it} J \delta^{it} J, \qquad \nabla^{it} = P^{it} \hat{J} \hat{\delta}^{it} \hat{J}.$$

Next, we comment on the classical case. We see that the dual group is given by the group von Neumann algebra which for abelian groups is isomorphic to the essentially bounded functions on the dual group. The multiplicative unitary corresponds to the left regular representation, which clarifies why it determines the structure of both a the quantum group and its dual.

Example 1.2.4. Let $G \to B(L^2(G)) : x \mapsto \lambda_x$ be the left regular representation. For $(M, \Delta) = (L^\infty(G), \Delta_G)$ as in Example 1.1.2, one finds that $\mathcal{H} \otimes \mathcal{H} = L^2(G) \otimes L^2(G) \simeq L^2(G \times G)$ and

$$(Wh)(x, y) = h(x, x^{-1}y).$$

For $f \in L^1(G)$, let ω_f be the functional on $L^\infty(G)$ defined by

$$\omega_f(g) = \int_G f(x)g(x)d_l x.$$

Then,

$$\lambda(\omega_f) = (\omega_f \otimes \iota)(W) = \int_G f(x)\lambda_x d_l x,$$

where the integral is in the σ-strong-$*$ topology. So λ is the left regular representation. We find that $\hat{M}$ is given by the group von Neumann algebra $\hat{M} = \mathcal{L}(G)$.

The coproduct is determined by the formula $\hat{\Delta}(\lambda_x) = \lambda_x \otimes \lambda_x$. The dual left Haar weight is given by the Plancherel weight [75]. For a continous, compactly supported function f on G, one finds $\hat{\varphi}(\lambda(f)) = f(e)$, where e is the identity of G.

If G is abelian, conjugation with the L^2-Fourier transform shows that this structure is isomorphic to $(L^\infty(\hat{G}), \Delta_{\hat{G}})$. So indeed, the quantum group dual is isomorphic to the Pontrjagin dual for locally compact abelian groups.

1.3 Quantum subgroups

The definition of a quantum subgroup is slightly delicate. For compact quantum groups the natural definition would be the following.

Definition 1.3.1. Let (A, Δ), (A_1, Δ_1) be compact quantum groups in the sense of Woronowicz. A surjective map $\pi : A \to A_1$ such that $(\pi \otimes \pi)\Delta = \Delta_1 \pi$ identifies (A_1, Δ_1) as a closed quantum subgroup of (A, Δ).

However, such a surjective map does not exist on the von Neumann algebraic level. And even on the C*-algebraic quantum groups we encounter later such a definition would not detect the closedness of a subgroup. The proper definition is motivated by the following result, which can be found in [25].

Proposition 1.3.2 (Proposition 5.2.8 of [25])**.** *Let K and G be locally compact groups and suppose that $\rho : K \to G$ is a continuous group homomorphism. The map $\mathcal{L}(K) \to \mathcal{L}(G) : \lambda_k \mapsto \lambda_{\rho(k)}$ extends to a faithful, normal, unital homomorphism if and only if $\rho : K \to G$ identifies K as a closed subgroup of G.*

In the quantum context, we borrow the following definition from [85].

Definition 1.3.3. Let (M, Δ) and (M_1, Δ_1) be locally compact quantum groups. Suppose that there exists an injective homomorphism $\hat{\pi} : \hat{M}_1 \to \hat{M}$ such that $(\hat{\pi} \otimes \hat{\pi})\hat{\Delta}_1 = \hat{\Delta}\hat{\pi}$. In that case, we say that $\hat{\pi}$ identifies (M_1, Δ_1) as a *closed quantum subgroup* of (M, Δ).

Note that our notation is dual to [85] since (M_1, Δ_1) and $(\hat{M}_1, \hat{\Delta}_1)$ are interchanged. We mention that Definitions 1.3.1 and 1.3.3 are equivalent under the equivalence of compact von Neumann algebraic quantum groups and Woronowicz C*-algebraic compact quantum groups.

1.4 C*-algebraic quantum groups

Kustermans and Vaes study quantum groups on three levels; the von Neumann algebraic and the reduced and universal C*-algebraic level. The situation is comparable to the study of group algebras where one has a group von Neumann algebra and a reduced and universal group C*-algebra. There is a passage to go from one level to the other on the level of objects which we describe here.

On the level of morphisms, which is quite a delicate study, there is no such passage. This feels a bit like a drawback, but in fact it can be used as an advantage since the morphisms on the various levels can enhance each other, see [46] for a good example.

Definition 1.4.1. A *C*-algebraic quantum group* (A, Δ) consists of the following data:

1. A C*-algebra A;
2. A non-degenerate homomorphism $\Delta : A \to M(A \otimes A)$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
3. Two approximate KMS-weights φ and ψ on A, such that

$$\varphi((\omega \otimes \iota)\Delta(x)) = \omega(1)\varphi(x), \qquad \omega \in A^*,\ a \in \mathfrak{m}_\varphi^+$$
$$\psi((\iota \otimes \omega)\Delta(x)) = \omega(1)\psi(x), \qquad a \in A^*,\ \omega \in \mathfrak{m}_\psi^+.$$

   We refer to Appendix A.3 for the definition of an approximate KMS-weight.

Again, we suppress the Haar weights in the notation. A C*-algebraic quantum group is *reduced* if the Haar weights are faithful.

Example 1.4.2. The reduced group C*-algebra with the restriction of the comultiplication and Haar weight defined in Example 1.2.4 forms a C*-algebraic quantum group which is reduced. Similarly, the universal group C*-algebra can be equiped with the structure of a C*-algebraic quantum group.

To every von Neumann algebraic locally compact quantum group (M, Δ), one can associate a reduced quantum group in the following way. One defines the C*-algebra M_c as the norm closure of $\{(\iota \otimes \omega)(W) \mid \omega \in \hat{M}_*\}$. Moreover, the reduced comultiplication is the restriction of Δ and is a well-defined map $\Delta_c : M_c \to M(M_c \otimes M_c)$. The Haar weights can be defined by restriction of φ and ψ, the Haar weights on M. Similarly, a dual reduced quantum group can be defined.

In [56] *universal* C*-algebraic quantum groups are defined. Instead of giving its definition, we show how to obtain a universal C*-algebraic quantum group from a von Neumann algebraic quantum group. Since every universal quantum group can be obtained in this way, one can take this as a definition.

Recall the Banach-$*$-algebra $M_*^\sharp$ from (1.4). Define a norm on this set in the following way:

$$\|\omega\|_u = \sup\{\|\pi(\omega)\| \mid \pi \text{ is a representation of } M_*^\sharp\}. \tag{1.6}$$

The map $\lambda : M_*^\sharp \to \hat{M}$ is an injective representation so that (1.6) a norm and not merely a semi-norm. Define $\hat{M}_u$ as the completion of $M_*^\sharp$ with respect to this norm. We denote

$$\lambda_u : M_*^\sharp \to \hat{M}_u$$

for the canonical inclusion. This C*-algebra $\hat{M}_u$ satisfies the following universal property. Any representation $\pi : M_*^\sharp \to B(\mathcal{K})$ lifts to a representation $\rho : \hat{M}_u \to B(\mathcal{K})$, i.e. $\rho\lambda_u = \pi$. By this universal property the representation $\lambda : M_*^\sharp \to \hat{M}_c$ lifts to a canonical surjective map

$$\hat{\vartheta} : \hat{M}_u \to \hat{M}_c.$$

One defines weights on $\hat{M}_u$ by setting $\hat{\varphi}_u = \hat{\varphi}_c\hat{\vartheta}$ and $\hat{\psi}_u = \hat{\psi}_c\hat{\vartheta}$. In [56], it is proved that $\hat{M}_u$ can be equipped with a comultiplication $\hat{\Delta}_u : \hat{M}_u \to M(\hat{M}_u \otimes \hat{M}_u)$ such that $(\hat{M}_u, \hat{\Delta}_u)$ is a universal C*-algebraic quantum group with Haar weights $\hat{\varphi}_u$ and $\hat{\psi}_u$.

Theorem 1.4.3 (Proposition 4.2 of [56]). *There exists a unique unitary element $\hat{\mathcal{V}} \in M(M_c \otimes \hat{M}_u)$ such that $(\Delta_c \otimes \iota)(\hat{\mathcal{V}}) = \hat{\mathcal{V}}_{13}\hat{\mathcal{V}}_{23}$ and $(\iota \otimes \hat{\vartheta})(\hat{\mathcal{V}}) = W$. Moreover, $(\omega \otimes \iota)(\hat{\mathcal{V}}) = \lambda_u(\omega)$ for every $\omega \in M_*^\sharp$.*

To every C*-algebraic quantum group, there is a canonical way to associate a reduced C*-algebraic quantum group to it, see [59, Remark 4.4]. Let (A, Δ_A) be a C*-algebraic quantum group. Let $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the GNS-representation with respect to the left Haar weight. Set the C*-algebra $A_c = \pi_\varphi(A)$. There is unique comultiplication $\Delta_c : A_c \to M(A_c \otimes A_c)$ such that $(\pi_\varphi \otimes \pi_\varphi)\Delta_A = \Delta_c\pi_\varphi$. Moreover, let $\tilde{\varphi}$ be the W*-lift of the weight φ, which is a weight on A_c''. Let φ_c be the restriction of $\tilde{\varphi}$ to A_c. Similarly, define ψ_c as the restriction of the W*-lift $\tilde{\psi}$ of ψ to A_c. Then (A_c, Δ_c) together with the weigths φ_c and ψ_c is a reduced

C^*-algebraic quantum group. Applying this construction to $(\hat{M}_u, \hat{\Delta}_u)$ will give $(\hat{M}_c, \hat{\Delta}_c)$.

Furthermore, set $M = A_c''$. Then, the comultiplication Δ_c constructed in the previous paragraph extends in a unique way to a unital and normal $*$-homomorphism $\Delta : M \to M \otimes M$. This way, (M, Δ) together with the weights $\tilde{\varphi}$ and $\tilde{\psi}$ is a von Neumann algebraic quantum group. We refer to Appendix A.3 for the definition of the W^*-lifts $\tilde{\varphi}$ and $\tilde{\psi}$.

1.5 Corepresentation theory

Let (M, Δ) be a locally compact quantum group.

Definition 1.5.1. A (unitary) *corepresentation* U on a Hilbert space $\mathcal{K}$ is a unitary $U \in M \otimes B(\mathcal{K})$ that satisfies $(\Delta \otimes \iota)(U) = U_{13}U_{23}$.

In principle, one is able to consider non-unitary corepresentations as well. However, in this thesis we make the assumption that every corepresentation is unitary unless explicitly stated otherwise. A corepresentation is called *finite dimensional* if the Hilbert space is finite dimensional. Using the pentagon equation and (1.1), we find that the mulitplicative unitary is corepresentation:

$$(\Delta \otimes \iota)(W) = W_{12}^* W_{23} W_{12} = W_{13} W_{23}.$$

Next, we comment on the classical setting.

Example 1.5.2. Let $(M, \Delta) = (L^\infty(G), \Delta_G)$, see Example 1.1.2. If π is a strongly continuous unitary representation $\pi : G \to B(\mathcal{K})$, then one can consider this mapping as an element of $L^\infty(G, B(\mathcal{K})) \simeq L^\infty(G) \otimes B(\mathcal{K})$. Denote the corresponding element by U_π. Then U_π is a corepresentation as the property $(\Delta \otimes \iota)(U_\pi) = (U_\pi)_{13}(U_\pi)_{23}$ follows from the homomorphism property of π. Conversely, every corepresentation of (M, Δ) is of this form.

In Example 1.5.2 the mulitplicative unitary corresponds to the left regular representation. Therefore, W is sometimes called the *left regular corepresentation*.

Definition 1.5.3. Two corepresentations $U_1 \in M \otimes B(\mathcal{K}_1), U_2 \in M \otimes B(\mathcal{K}_2)$ are called *equivalent* if there is a unitary map $T : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$(1 \otimes T)U_1 = U_2(1 \otimes T).$$

In that case, T is called a (unitary) *intertwiner*.

It is straightforward to check that equivalence of corepresentations is an equivalence relation. We denote the equivalence classes of irreducible corepresentations of (M, Δ) by $\mathrm{IC}(M)$. For convenience of notation we will sometimes simply write $U \in \mathrm{IC}(M)$ to say that U is an irreducible corepresentation of (M, Δ) instead of considering its equivalence class.

By considering functionals $\omega_{\xi,\eta} \in M_*$, one easily proves the following lemma.

Lemma 1.5.4 (Lemma 1.2.3 of [19]). *Consider corepresentations $U_1 \in M \otimes B(\mathcal{K}_1), U_2 \in M \otimes B(\mathcal{K}_2)$. U_1 and U_2 are equivalent if and only if there is a unitary map $T : \mathcal{K}_1 \to \mathcal{K}_2$ such that $T(\omega \otimes \iota)(U_1) = (\omega \otimes \iota)(U_2)T$ for all $\omega \in M_*$.*

The following lemma is more difficult to prove. The result relies on the technique used in [86, Proposition 5.6].

Lemma 1.5.5 (Lemma 1.2.2 of [19]). *Let $U \in M \otimes B(\mathcal{K})$ be a corepresentation. Then, for $\theta \in B(\mathcal{K})_*$, we have $(\iota \otimes \theta)(U) \in Dom(S)$ and $S(\iota \otimes \theta)(U) = (\iota \otimes \theta)(U^*)$. In particular, if $\omega \in M_*^\sharp$, then $(\omega^* \otimes \iota)(U) = (\omega \otimes \iota)(U)^*$.*

Proof. For the first statement, we refer to [19, Lemma 1.2.2]. Then, for $\theta \in B(\mathcal{K})_*$,

$$\theta(\omega^* \otimes \iota)(U) = \overline{\omega}S((\iota \otimes \theta)(U)) = \overline{\omega}(\iota \otimes \theta)(U^*) = \theta(\overline{\omega} \otimes \iota)(U^*) = \theta((\omega \otimes \iota)(U)^*).$$

□

Now, we turn to decomposition theory for corepresentations. Let $U \in M \otimes B(\mathcal{K})$ be a corepresentation. A closed subspace $\mathcal{L} \subseteq \mathcal{K}$ is called *invariant* if $(\omega \otimes \iota)(U)\mathcal{L} \subseteq \mathcal{L}$ for all $\omega \in M_*$. U is called *irreducible* if it has no closed invariant subspaces except for $\{0\}$ and $\mathcal{K}$.

Lemma 1.5.6. *Let $\mathcal{L} \subseteq \mathcal{K}$ be a closed invariant subspace for a corepresentation $U \in M \otimes B(\mathcal{K})$. Then, $\mathcal{L}^\perp$ is an invariant subspace as well.*

Proof. Let $v \in \mathcal{L}, w \in \mathcal{L}^\perp$. For $\omega \in M_*^\sharp$, we have $(\omega^* \otimes \iota)(U)v \in \mathcal{L}$, so using Lemma 1.5.5,

$$\langle(\omega \otimes \iota)(U)w, v\rangle = \langle w, (\omega \otimes \iota)(U)^*v\rangle = \langle w, (\omega^* \otimes \iota)(U)v\rangle = 0.$$

Using the fact that $M_*^\sharp$ is dense in M_*, see Lemma 1.1.5, we find $(\omega \otimes \iota)(U)w \in \mathcal{L}^\perp$. □

The next definition basically says that a direct integral of corepresentations is a direct intregral as operators. In particular, direct integral decompositions of the multiplicative unitary play a central role in this thesis.

Definition 1.5.7. Let (X, μ) be a standard measure space. For every $x \in X$, let $U_x \in M \otimes B(\mathcal{K}_x)$ be a corepresentation. Then, $(U_x)_{x \in X}$ is called a *μ-measurable field of corepresentations* if $(U_x)_{x \in X}$ is a measurable field of operators. In that case the direct integral

$$\int_X^\oplus U_x d\mu(x) \in M \otimes B(\int_X^\oplus \mathcal{K}_x d\mu(x))$$

is a corepresentation.

In particular the previous definition defines direct sums. Suppose that $\mathcal{L}$ is an invariant subspace for a corepresentation $U \in M \otimes B(\mathcal{K})$. Let U_1 be the restriction of U to the space $\mathcal{H} \otimes \mathcal{L}$ and let U_2 be the restriction of U to $\mathcal{H} \otimes \mathcal{L}^{\perp}$, c.f. Lemma 1.5.6. Then U is equivalent to $U_1 \oplus U_2$. For compact quantum groups, there is a Peter-Weyl theorem proved by Woronowicz.

Theorem 1.5.8 ([97]). *Let (M, Δ) be a compact quantum group. Every irreducible corepresentation of (M, Δ) is finite dimensional. Furthermore,*

$$W \simeq \bigoplus_{U \in \mathrm{IC}(M)} U \otimes 1_{\dim(\mathcal{H}_U)},$$

where $\mathcal{H}_U$ is the corepresentation space of U. Moreover, for every $U \in \mathrm{IC}(M)$, there is a strictly positive operator $D_U \in B(\mathcal{H}_U)$ such that the following orthogonality relation hold. For $\xi, \xi', \eta, \eta' \in \mathcal{H}_U$, we have

$$\varphi\left((\iota \otimes \omega_{\xi,\eta})(U)^*(\iota \otimes \omega_{\xi',\eta'})(U)\right) = \langle D_U \eta, D_U \eta' \rangle \langle \xi', \xi \rangle.$$

Furthermore, suppose that $U, U' \in \mathrm{IC}(M)$ such that U and U' are non-equivalent. For $\xi, \xi', \eta, \eta' \in \mathcal{H}_U$, we have

$$\varphi\left((\iota \otimes \omega_{\xi,\eta})(U)^*(\iota \otimes \omega_{\xi',\eta'})(U')\right) = 0.$$

It follows that the dual of a compact quantum group is spatially isomorphic to a direct sum of *finite dimensional* matrix algebras. The dual of a compact quantum group is also called a *discrete quantum group*.

Let $U \in M \otimes B(\mathcal{K})$ be a corepresentation of (M, Δ). Then we get a representation of $M_*^\sharp \to B(\mathcal{K}) : \omega \mapsto (\omega \otimes \iota)(U)$. Indeed, for $\omega_1, \omega_2 \in M_*$,

$$\begin{aligned}(\omega_1 * \omega_2 \otimes \iota)(U) &= (\omega_1 \otimes \omega_2 \otimes \iota)(\Delta \otimes \iota)(U) \\ &= (\omega_1 \otimes \omega_2 \otimes \iota) U_{13} U_{23} = (\omega_1 \otimes \iota)(U)(\omega_2 \otimes \iota)(U),\end{aligned}$$

and using Lemma 1.5.5, for $\omega \in M_*^\sharp$, $(\omega^* \otimes \iota)(U) = (\omega \otimes \iota)(U)^*$. In fact, in [56] it is proved that there is a 1-1 correspondence between corepresentations of (M, Δ) and representations of $\hat{M}_u$. We reformulate this result in the following theorem.

Theorem 1.5.9 (Corollary 4.3 of [56]). *For every non-degenerate representation π of $\hat{M}_u$ on a Hilbert space $\mathcal{K}_\pi$, there exists a corepresentation $U_\pi \in M \otimes B(\mathcal{K}_\pi)$ unique up to equivalence such that π is given by*

$$\pi : \lambda_u(\omega) \mapsto (\omega \otimes \iota)(U_\pi), \qquad \omega \in M_*^\sharp.$$

Moreover, U_π is given by $(\iota \otimes \pi)(\hat{\mathcal{V}})$.

Proof. Let $A \subseteq B(\mathcal{K})$ be a C*-algebra acting non-degenerately on $\mathcal{K}$. Then $M(A)$ is a C*-subalgebra of $B(\mathcal{K})$ by [62, Proposition 2.1]. Moreover, $M(A)$ is contained in the strong closure of A. Indeed, let $x \in M(A)$ and let $(e_j)_{j \in J}$ be an approximate unit for A. We find that for $a \in A, \xi \in \mathcal{K}$, we have $xe_j a\xi \to xa\xi$ in norm. By non-degeneracy of A and since $(xe_j)_{j\in J}$ is bounded, we get $xe_j \to x$ strongly. We conclude that

$$M(M_c \otimes B(\mathcal{K}_\pi)) \subseteq (M_c \otimes B(\mathcal{K}_\pi))'' \subseteq (M_c'' \otimes B(\mathcal{K}_\pi)'')'' = M \otimes B(\mathcal{K}_\pi).$$

Here, the first and second tensor products are the injective tensor product, whereas the others are von Neumann algebraic tensor products.

Now, according to [56, Corollary 4.3], $U_\pi = (\iota \otimes \pi)(\hat{\mathcal{V}}) \in M(M_c \otimes B(\mathcal{K}_\pi))$ satisfies all the desired properties. The uniqueness statement follows since $M_*^\sharp$ is dense in M_*. □

From Lemma 1.5.4 it follows that the 1-1 correspondence given in Theorem 1.5.9 descends to a correspondence between the equivalence classes of (co)reprepresentations. Furthermore, it is clear that in Theorem 1.5.9 a closed subspace $\mathcal{L} \subseteq \mathcal{K}_\pi$ is invariant for π if and only if it is invariant for U_π. Moreover, if we have a direct integral decomposition $\pi = \int_X^\oplus \pi_x d\mu(x)$, then $U_\pi = \int_X^\oplus U_{\pi_x} d\mu(x)$.

1.6 Plancherel theorems

In [19] Desmedt proved a decomposition theorem for the multiplicative unitary associated with a quantum group. Here we state the theorem as well as some modifications and corollaries. These serve as a preparation for Chapter 2. We need to refer to the proof of the Plancherel theorem. Since [19] is not freely available, we give a sketch of Desmedt's proof here.

Desmedt's decomposition theorem can be considered as the quantum analogue of the Plancherel theorem for non-unimodular groups as proved by Duflo and Moore [24]. Here, they give a decomposition of the left regular representation of a group. The intertwiner is given by a formula in terms of a measurable field of strictly positive self-adjoint operators, called *Duflo-Moore* operators. The operators will play an important role in Chapter 2. Whereas for uni-modular groups the Duflo-Moore operators are trivial, for uni-modular quantum groups the Duflo-Moore operators do not have to be trivial.

On the other hand, the Plancherel theorem is an extension of the quantum Peter-Weyl Theorem 1.5.8 to the non-compact setting.

For the following theorem, recall that if $\mathcal{K}$ is a Hilbert space, then the Hilbert-Schmidt operators $\mathrm{B}_2(\mathcal{K})$ can be identified with $\mathcal{K} \otimes \overline{\mathcal{K}}$ by the identification $\theta_{\xi,\eta} \mapsto \xi \otimes \overline{\eta}$. We use bars to denote the elements of the conjugate Hilbert space $\overline{\mathcal{K}}$.

Theorem 1.6.1 (Theorem 3.4.1 of [19]). *Let (M, Δ) be a locally compact quantum group such that $\hat{M}$ is a type I von Neumann algebra and such that $\hat{M}_u$*

is a separable C^*-algebra. There exists a standard measure μ on $\mathrm{IC}(M)$, a μ-measurable field $(\mathcal{H}_U)_{U\in\mathrm{IC}(M)}$ of Hilbert spaces, a measurable field $(D_U)_{U\in\mathrm{IC}(M)}$ of self-adjoint, strictly positive operators and an isomorphism

$$\mathcal{Q}_L : \mathcal{H} \to \int_{\mathrm{IC}(M)}^{\oplus} B_2(\mathcal{H}_U) d\mu(U),$$

with the following properties:

1. *For all $\alpha \in \mathcal{I}$ and μ-almost all $U \in \mathrm{IC}(M)$, the operator $(\alpha \otimes \iota)(U)D_U^{-1}$ is bounded and $(\alpha \otimes \iota)(U) \cdot D_U^{-1}$ is a Hilbert-Schmidt operator on $\mathcal{H}_U$.*
2. *For all $\alpha, \beta \in \mathcal{I}$ one has the Parseval formula*
$$\langle \xi(\alpha), \xi(\beta) \rangle = \int_{\mathrm{IC}(M)} \mathrm{Tr}\left(\left((\beta \otimes \iota)(U) \cdot D_U^{-1} \right)^* \left((\alpha \otimes \iota)(U) \cdot D_U^{-1} \right) \right) d\mu(U),$$
and $\mathcal{Q}_L$ is the isometric extension of
$$\hat{\Lambda}(\lambda(\mathcal{I})) \to \int_{\mathrm{IC}(M)}^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}}_U d\mu(U) : \ \xi(\alpha) \mapsto \int_{\mathrm{IC}(M)}^{\oplus} (\alpha \otimes \iota)(U) \cdot D_U^{-1} d\mu(U).$$
3. *$\mathcal{Q}_L$ satisfies the following intertwining property:*
$$\mathcal{Q}_L(\omega \otimes \iota)(W) = \int_{\mathrm{IC}(M)}^{\oplus} (\omega \otimes \iota)(U) \otimes 1_{\overline{\mathcal{H}}_U} d\mu(U) \mathcal{Q}_L.$$
4. *Suppose that we have a standard measure μ' and there exists μ'-measurable fields $(\mathcal{H}'_U)_{U\in\mathrm{IC}(M)}$ and $(D'_U)_{U\in\mathrm{IC}(M)}$ that satisfy the same properties, then μ and μ' are equivalent and for μ-almost every $U \in \mathrm{IC}(M)$ we have*
$$D'_U = \frac{d\mu}{d\mu'}(U) T_U D_U T_U^{-1},$$
with $T_U : \mathcal{H}_U \to \mathcal{H}'_U$ a an intertwiner for μ'-almost every $U \in \mathrm{IC}(M)$.

The operators D_U appearing in the intertwiner of the previous theorem are called *Duflo-Moore* operators.

This theorem is only part of the result obtained in [19]. We give a sketch of its proof. The theorem is basically obtained as a corollary of the following auxiliary result.

Theorem 1.6.2 (Theorem 3.4.5 of [19]). *Let A be a separable C^*-algebra with lower semi-continuous, densely defined, approximately KMS-weight ϕ such that $\pi_\phi(A)''$ is a type I von Neumann algebra. Then, there exists a positive measure μ on $\mathrm{IR}(A)$, a μ-measurable field of Hilbert spaces $(\mathcal{K}_\sigma)_{\sigma\in\mathrm{IR}(A)}$, a μ-measurable field $(\pi_\sigma)_{\sigma\in\mathrm{IR}(A)}$ of representations of A on $\mathcal{K}_\sigma$ such that π_σ belongs to the class σ for every $\sigma \in \mathrm{IR}(A)$, a measurable field $(D_\sigma)_{\sigma\in\mathrm{IR}(A)}$ of self-adjoint, strictly positive operators and an isomorphism $\mathcal{P}$ of the GNS-space $\mathcal{H}_\phi$ onto $\int_{\mathrm{IR}(A)}^{\oplus} \mathcal{K}_\sigma \otimes \overline{\mathcal{K}_\sigma} d\mu(\sigma)$ with the following properties:*

1. *For all $x \in \mathfrak{n}_\phi$ and μ-almost all $\sigma \in \mathrm{IR}(A)$, the operator $\pi_\sigma(x)D_\sigma^{-1}$ is bounded and $\pi_\sigma(x) \cdot D_\sigma^{-1}$ is Hilbert-Schmidt.*

2. *For all $a, b \in \mathfrak{n}_\phi$ one has the Parseval formula*

$$\langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle = \int_{\mathrm{IR}(A)} \mathrm{Tr}\left((\pi_\sigma(b) \cdot D_\sigma^{-1})^* (\pi_\sigma(a) \cdot D_\sigma^{-1}) \right) d\mu(\sigma),$$

and $\mathcal{P}$ is the isometric extension of

$$\Lambda_\phi(\mathfrak{n}_\phi) \to \int_{\mathrm{IR}(A)}^{\oplus} \mathcal{K}_\sigma \otimes \overline{\mathcal{K}_\sigma} d\mu(\sigma) : \ \Lambda_\phi(x) \mapsto \int_{\mathrm{IR}(A)}^{\oplus} \pi_\sigma(x) \cdot D_\sigma^{-1} d\mu(\sigma).$$

3. *$\mathcal{P}$ intertwines the GNS-representation $\pi_\phi : A \to B(\mathcal{H}_\phi)$ with the direct integral representation of A given by*

$$\int_{\mathrm{IR}(A)}^{\oplus} \pi_\sigma \otimes 1_{\overline{\mathcal{K}}_\sigma} d\mu(\sigma).$$

Sketch of the proof following [19]. Since $\pi_\phi(A)''$ is type I, it is of the form

$$\pi_\phi(A)'' \simeq \int_X^{\oplus} B(\mathcal{K}_x) d\mu(x),$$

where μ is a measure on the standard measure space X and $(\mathcal{K}_x)_{x \in X}$ is a field of separable Hilbert spaces [74, Ch V, Theorem 1.27]. Therefore, $\pi_\phi(A)''$ has a canonical trace $\tau = \oplus_n (\mathrm{Tr}_n \otimes \int_{X_n} d\mu)$, where X_n contains all $x \in X$ such that $\mathcal{K}_x$ is of dimension n and Tr_n is the trace on the n-dimensional Hilbert space ($n = \infty$ is allowed). Note that the GNS-space with respect to τ is given by $\int_X^{\oplus} \mathcal{K}_x \otimes \overline{\mathcal{K}}_x d\mu(x)$ and $\mathfrak{n}_\tau$ equals the Hilbert-Schmidt operators in $\pi_\phi(A)''$.

Let $\tilde{\phi}$ be the W*-lift of ϕ, which is a normal, semi-finite, faithful weight since we assumed ϕ to be an approximately KMS weight, see Appendix A.3. Now, by Theorem A.4.2 or the weaker result [75, Theorem 3.14], there is a self-adjoint strictly positive operator D which is affiliated with $\pi_\phi(A)''$ such

$$\tilde{\phi} = \tau_{D^{-2}}. \tag{1.7}$$

That is, D^{-2} is the Radon-Nikodym derivative of $\tilde{\phi}$ with respect to τ.

By [61, Theorem 1.8], we see that D has a direct integral decomposition $D = \int_X^{\oplus} D_x d\mu(x)$. Let $\mathfrak{n}_{\tilde{\phi}}^0$ denote the set of $y \in \pi_\phi(A)''$ such that yD^{-1} is bounded and $y \cdot D^{-1} \in \mathfrak{n}_\tau$. From the first section of [82] one finds that $\mathfrak{n}_{\tilde{\phi}}^0$ is a core for $\Lambda_{\tilde{\phi}}$. Hence, we have $\Lambda_{\tilde{\phi}}(y) = \Lambda_\tau(y \cdot D^{-1}) = \int_X^{\oplus} y_x \cdot D_x^{-1} d\mu(x)$. So for $y \in \mathfrak{n}_{\tilde{\phi}}^0$ the map

$$\Lambda_{\tilde{\phi}}(y) \mapsto \int_X^{\oplus} y_x \cdot D_x^{-1} d\mu(x),$$

is an isometry. Desmedt proves that actually for any $y \in \mathfrak{n}_{\tilde{\phi}}$ one has that for almost all $x \in X$, the operator $y_x D_x^{-1}$ is bounded and $y_x \cdot D_x^{-1}$ is Hilbert-Schmidt. Furthermore, for all $y \in \pi_\phi(A)''$,

$$y = \pi_{\tilde{\phi}}(y) = \pi_\tau(y) = \int_X^\oplus \pi_{\mathrm{Tr}_x}(y_x) d\mu(x) = \int_X^\oplus (y_x \otimes 1) d\mu(x), \tag{1.8}$$

where Tr_x is the unique normal, semi-finite, faithful trace on $B(\mathcal{H}_x)$. Since $\pi_\phi(A)$ commutes with the diagonalizable operators, we find from [74, Theorem IV.8.25] that there exists a measurable field of representations $(\zeta_x)_{x\in X}$ of A on $\mathcal{K}_x \otimes \overline{\mathcal{K}}_x$ such that $\pi_\phi = \int_X^\oplus \zeta_x d\mu(x)$, and by (1.8) this implies that there exists a measurable field of representations π_x of A on $\mathcal{K}_x$ such that, for all $a \in A$, $\zeta_x(a) = \pi_x(a) \otimes 1$ for almost all $x \in X$. By the decomposition of $\pi_\phi(A)''$ we find that $\pi_x(A)'' = B(\mathcal{K}_x)$, hence $\pi_x(A)' = \mathbb{C}1$ for almost all $x \in X$. Therefore, almost all π_x are irreducible and [22, Lemma 8.4.1(iii)] yields non-equivalence of almost all π_x. Substracting a negligible set from X we may assume that π_x are non-equivalent and by [22, Proposition 8.1.8], it follows that $x \mapsto \pi_x$ is a Borel map from X to $\mathrm{IR}(A)$ which is injective. We extend μ under this correspondence to the whole $\mathrm{IR}(A)$, which proves the auxiliary theorem. □

Now, Theorem 1.6.1 can be proved by considering $A = \hat{M}_u$ and using the correspondence between $\mathrm{IR}(\hat{M}_u)$ and $\mathrm{IC}(M)$, see Theorem 1.5.9. This defines μ as a measure on $\mathrm{IC}(M)$. Using the following relations following from Theorem 1.5.9, one easily transfers Theorem 1.6.2 (1) - (3) to Theorem 1.6.1 (1) - (3):

$$\begin{aligned} \pi\left((\omega\otimes\iota)(\hat{\mathcal{V}})\right) &= (\omega\otimes\iota)(U_\pi), && \text{for } \pi \in \mathrm{IR}(\hat{M}_u); \\ \xi(\omega) &= \hat{\Lambda}\left((\omega\otimes\iota)(W)\right) = \hat{\Lambda}\left((\omega\otimes\iota)(\iota\otimes\hat{\vartheta})(\hat{\mathcal{V}})\right) && \\ &= \hat{\Lambda}\hat{\vartheta}\left((\omega\otimes\iota)(\hat{\mathcal{V}})\right) = \hat{\Lambda}_u\left((\omega\otimes\iota)(\hat{\mathcal{V}})\right), && \text{for } \omega \in \mathcal{I}. \end{aligned} \tag{1.9}$$

Property (4), relies on [22, Proposition 8.2.4], see [19] for the complete proof.

Here μ is called the left Plancherel measure and $\mathcal{Q}_L$ is called the left Plancherel transform. We deal with the following analogue of the above theorem as well. We state only the relevant part.

Theorem 1.6.3 (Remark 3.4.11 of [19]). *Let (M,Δ) be a locally compact quantum group such that $\hat{M}$ is a type I von Neumann algebra and such that $\hat{M}_u$ is a separable C*-algebra. There exist a standard measure μ_R on $\mathrm{IC}(M)$, a μ_R-measurable field $(\mathcal{K}_U)_{U\in\mathrm{IC}(M)}$ of Hilbert spaces, a μ_R-measurable field $(E_U)_{U\in\mathrm{IC}(M)}$ of self-adjoint, strictly positive operators and an isomorphism*

$$\mathcal{Q}_R : \mathcal{H} \to \int_{\mathrm{IC}(M)}^\oplus B_2(\mathcal{K}_U) d\nu(U),$$

with the following properties:

1. *For all $\alpha \in \mathcal{I}_R$ and μ_R-almost all $U \in \mathrm{IC}(M)$, the operator $(\alpha \otimes \iota)(U)E_U^{-1}$ is bounded and $(\alpha \otimes \iota)(U) \cdot E_U^{-1}$ is a Hilbert-Schmidt operator on $\mathcal{K}_U$.*

2. *For all $\alpha, \beta \in \mathcal{I}_R$ one has the Parseval formula*

$$\langle \xi_R(\overline{\alpha^*}), \xi_R(\overline{\beta^*}) \rangle = \int_{\mathrm{IC}(M)} \mathrm{Tr}\left(\left((\beta \otimes \iota)(U) \cdot E_U^{-1}\right)^* \left((\alpha \otimes \iota)(U) \cdot E_U^{-1}\right)\right) d\mu_R(U),$$

and $\mathcal{Q}_R$ is the isometric extension of

$$\hat{\Gamma}(\lambda(\mathcal{I})) \to \int_{\mathrm{IC}(M)}^{\oplus} B_2(H_U) d\mu_R(U) : \;\; \xi_R(\overline{\alpha^*}) \mapsto \int_{\mathrm{IC}(M)}^{\oplus} (\alpha \otimes \iota)(U) \cdot E_U^{-1} d\mu_R(U).$$

3. *The measure μ_R can be choosen equal to the measure μ of Theorem 1.6.1 and the μ_R-measurable field of Hilbert spaces $(\mathcal{K}_U)_U$ can be choosen equal to $(\mathcal{H}_U)_U$, the measurable field of Hilbert spaces of Theorem 1.6.1.*

The theorem can be obtained from the auxiliary Theorem 1.6.2 and using the relations between the right Haar weights ψ and the right Haar weight $\hat{\psi}_u$ on the universal dual quantum group.

We elaborate on the third statement. In order to keep the argument more symmetric, write for a moment $\pi_{\hat{\varphi}_u}$ for $\hat{\pi}_u$, the GNS-representation of $\hat{\varphi}_u$. Write $\pi_{\hat{\varphi}}$ for $\hat{\pi}$, the GNS-representation for $\hat{\varphi}$. Let $\pi_{\hat{\psi}_u}$ and $\pi_{\hat{\psi}}$ be GNS-representations for $\hat{\psi}_u$ and $\hat{\psi}$ respectively.

Since $\hat{\varphi}_u$ and $\hat{\psi}_u$ are both approximately KMS-weights on the universal dual $\hat{M}_u$, their W*-lifts are normal, semi-finite, faithful weights, so that [75, Theorem VIII.3.2] implies that the representations π_φ and π_ψ are equivalent. Hence,

$$\pi_{\hat{\varphi}_u}(\hat{M}_u)'' = \pi_{\hat{\varphi}}(\hat{M}) \simeq \pi_{\hat{\psi}}(\hat{M}) = \pi_{\hat{\psi}_u}(\hat{M}_u)''. \tag{1.10}$$

The proof of Theorem 1.6.2 shows that the measures μ and μ_R together with the measurable fields of Hilbert spaces $(\mathcal{H}_U)_{U \in \mathrm{IC}(M)}$ and $(\mathcal{K}_U)_{U \in \mathrm{IC}(M)}$ in Theorems 1.6.1 and 1.6.3 arise from the direct integral decompositions of $\pi_{\hat{\varphi}_u}(\hat{M}_u)''$ and $\pi_{\hat{\psi}_u}(\hat{M}_u)''$, respectively. That is:

$$\pi_{\hat{\varphi}_u}(\hat{M}_u)'' = \int_X^{\oplus} B(\mathcal{H}_x) d\mu(x), \qquad \pi_{\hat{\psi}_u}(\hat{M}_u)'' = \int_Y^{\oplus} B(\mathcal{K}_y) d\mu_R(y).$$

By (1.10) we may assume that $\mu = \mu_R$, $X = Y$ and $(\mathcal{H}_x)_{x \in X} = (\mathcal{K}_x)_{x \in X}$. Furthermore, by (1.8) we have $\pi_{\hat{\varphi}}(y) = y = \pi_{\hat{\psi}}(y), \forall y \in \hat{M}$, which shows that the correspondence between X and the measurable subspace $\mathrm{IR}(\hat{M}_u)$ is the same in the proof of Theorem 1.6.2 for both $\phi = \hat{\varphi}_u$ and $\phi = \hat{\psi}_u$. This proves the third statement of Theorem 1.6.3.

Here μ_R is called the *right Plancherel measure* and $\mathcal{Q}_R$ is called the *right Plancherel transform*. Since μ_R may be chosen equal to μ, we will simply speak about the *Plancherel measure* μ, without specifying left and right. Similarly, we identify $(\mathcal{K}_U)_{U \in \mathrm{IC}(M)}$ with $(\mathcal{H}_U)_{U \in \mathrm{IC}(M)}$.

Remark 1.6.4. Theorems 1.6.1 and 1.6.3 remain valid when the assumption that $\hat{M}_u$ is separable (universal norm) is replaced by the assumption that $\hat{M}_c$ is separable (reduced norm) and the measure space $\mathrm{IC}(M)$ is replaced by the measure space $\mathrm{IR}(\hat{M}_c)$. The advantage of this, is that this condition is generally much easier to check.

The proof is a minor modification of the proof of Theorem 1.6.1. One applies the auxiliary Theorem 1.6.2 to $A = \hat{M}_c$ with $\phi = \hat{\varphi}_c$ (or for the right version $\hat{\psi}_c$). Instead of relations (1.9) one uses:

$$\begin{aligned} \pi\left((\omega \otimes \iota)(W)\right) &= \pi\left((\omega \otimes \iota)(\iota \otimes \hat{\vartheta})(\hat{\mathcal{V}})\right) = (\omega \otimes \iota)(U_{\pi\hat{\vartheta}}), && \text{for } \pi \in \mathrm{IR}(\hat{M}_c); \\ \xi(\omega) &= \hat{\Lambda}\left((\omega \otimes \iota)(W)\right), && \text{for } \omega \in \mathcal{I}. \end{aligned}$$

In [19, Theorem 3.4.8] Desmedt proves that the support of the Plancherel measure equals $\mathrm{IR}(\hat{M}_c)$, which is in agreement with this observation.

Remark 1.6.5. The corepresentations that appear as discrete mass points in the Plancherel measure correspond to the square integrable corepresentations in the sense of [5, Definition 3.2] or the equivalent definition of left square integrable corepresentations as in [19, Definition 3.2.29]. A proof of this can be found in [19, Theorem 3.4.10]. In particular, for compact quantum groups, every corepresentation is square integrable. Hence, the direct integral decomposition in the Plancherel theorem turns into a direct sum and we recover the Peter-Weyl theorem.

1.7 Preliminaries on $SU_q(2)$

We define $SU_q(2)$ in a von Neumann algebraic sense. We emphasize that this is not the best way for a first approach of $SU_q(2)$, since it can perfectly be understood in a purely algebraic setting. However, it is the operator algebraic approach that is more suitable here.

Definition 1.7.1 (of $SU_q(2)$). Let $0 < q < 1$. We set $\mathcal{H} = L^2(\mathbb{N}) \otimes L^2(\mathbb{T})$. Let $(e_i)_{i\in\mathbb{N}}$ be the canonical orthonormal basis of $L^2(\mathbb{N})$ and let $(f_k)_{k\in\mathbb{Z}}$ be the canonical orthonormal basis for $L^2(\mathbb{T})$ (so $f_k = \zeta^k$, where ζ is the identity function on the complex unit circle $\mathbb{T}$). Define operators α, γ given by:

$$\alpha\, e_i \otimes f_k = \sqrt{1 - q^{2i}} e_{i-1} \otimes f_k, \qquad \gamma\, e_i \otimes f_k = q^i e_i \otimes f_{k+1}. \tag{1.11}$$

Here, $e_{-1} = 0$. Then, α, γ satisfy the well-known relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1, \quad \gamma\gamma^* = \gamma^*\gamma, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*.$$

Let M be the von Neumann algebra generated by α and γ, i.e.

$$M = B(L^2(\mathbb{N})) \otimes L^\infty(\mathbb{T}) \simeq L^\infty(\mathbb{T}, B(L^2(\mathbb{N}))).$$

Let $x = x(t) \in L^\infty(\mathbb{T}, B(L^2(\mathbb{N})))$ and define a weight φ on M by the state:

$$\varphi(x) = \frac{(1-q^2)}{2\pi} \int_{\mathbb{T}} \sum_{i=0}^{\infty} q^{2i} \langle x(t) e_i, e_i \rangle dt.$$

Define a coproduct by the unique unital, normal $*$-homomorphism $\Delta : M \to M \otimes M$ extending

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

The pair (M, Δ) with Haar weight φ is a compact (uni-modular) quantum group called $SU_q(2)$ or *quantum* $SU(2)$.

We explicitly describe the Peter-Weyl decomposition for $SU_q(2)$. Let (M, Δ) be $SU_q(2)$. Recall [48] that for every $l \in \frac{1}{2}\mathbb{N}$, there exists a unique irreducible corepresentation $t^{(l)} \in M \otimes M_{2l+1}(\mathbb{C})$. Explicitly, they can be defined as follows. Consider $\mathbb{C}^{2l+1}$ as the vector space freely generated by homogeneous polynomials in α and γ of degree $2l$. So we have basis vectors

$$g_k^{(l)} = \begin{bmatrix} 2l \\ l-k \end{bmatrix}_{q^{-2}}^{\frac{1}{2}} \alpha^{l+k}\gamma^{l-k}, \qquad \text{with } k \in \{-l, -l+1, \ldots, l-1, l\}, \quad (1.12)$$

where

$$\begin{bmatrix} i \\ j \end{bmatrix}_q = \frac{(q;q)_i}{(q;q)_j (q;q)_{i-j}}, \qquad \text{and} \quad (a;q)_i = (1-a)(1-aq)\ldots(1-aq^{i-1}).$$

Then,

$$\Delta(g_k^{(l)}) \in M \otimes \mathrm{span}\{g_i^{(l)} \mid i \in \{-l, -l+1, \ldots, l-1, l\}\}.$$

So $\Delta : \mathbb{C}^{2l+1} \to M \otimes \mathbb{C}^{2l+1}$ can be considered as a linear map with coefficients in M, i.e. the restriction of Δ to $\mathrm{span}\{g_i^{(l)} \mid i \in \{-l, -l+1, \ldots, l-1, l\}\}$ is an element of $M \otimes M_{2l+1}(\mathbb{C})$ (sometimes explicitly called a *corepresentation matrix*). It is not hard to check that this defines a corepresentation for every $l \in \frac{1}{2}\mathbb{N}$.

Proposition 1.7.2 (Proposition 4.16 of [48]). *The corepresentations* $\{t^{(l)} \mid l \in \frac{1}{2}\mathbb{N}\}$ *form a maximal set of irreducible, unitary corepresentations.*

In fact, the Peter-Weyl decomposition becomes:

$$W \simeq \bigoplus_{l \in \frac{1}{2}\mathbb{N}} t^{(l)} \otimes 1_{2l+1} \quad \left(\in M \otimes \bigoplus_{l \in \mathbb{N}} M_{2l+1}(\mathbb{C}) \otimes M_{2l+1}(\mathbb{C}) \right).$$

So every corepresentation $t^{(l)}$ appears $2l+1$ times in the multiplicative unitary. Let us denote $t_{i,j}^{(l)}$ for the matrix elements $(\iota \otimes \omega_{g_j^{(l)}, g_i^{(l)}})(t^{(l)})$. Writing $D^{(l)}$ for $D_{t^{(l)}}$, the orthogonality relations of Theorem 1.5.8 can be written as:

$$\varphi((t_{i,j}^{(l)})^* t_{i',j'}^{(l')}) = \delta_{l,l'} \delta_{j,j'} \langle D^{(l)} g_i^{(l)}, D^{(l)} g_{i'}^{(l)} \rangle. \qquad (1.13)$$

It follows that $\hat{M} \simeq \oplus_{l\in\frac{1}{2}\mathbb{N}} M_{2l+1}(\mathbb{C})$. We put $D = \oplus_{l\in\frac{1}{2}\mathbb{N}} D^{(l)}$, so that D is affiliated with $\hat{M}$. Formally, one could define D as follows: first set $D^{it} = \oplus_{l\in\frac{1}{2}\mathbb{N}} (D^{(l)})^{it}$ and then let D be the generator of the one parameter group $t \mapsto D^{it}$. It is a well known consequence of (1.7) that

$$\hat{\sigma}_t(x) = D^{-2it} x D^{2it}, \qquad x \in \hat{M}. \tag{1.14}$$

The following two lemma follows directly by means of an explicit computation involving the corepresentation matrices $t^{(l)}$ described above.

Lemma 1.7.3. *Explicitly, for $n \in \mathbb{N}$ we have matrix coefficients*

$$t^{(n/2)}_{n/2,n/2} = \alpha^n, \qquad t^{(n/2)}_{-n/2,-n/2} = (\alpha^*)^n.$$

Lemma 1.7.4 (Theorem 4.17 of [48])**.** *For every $l \in \frac{1}{2}\mathbb{N}$, the matrix $D^{(l)}$ is diagonal with respect to the basis $g^{(l)}_k, k \in \{-l, -l+1, \ldots, l\}$.*

1.8 Preliminaries on $SU_q(1,1)_{\text{ext}}$

The quantum group $SU_q(1,1)_{\text{ext}}$ is considered as the q-deformation of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$. The quantum group was introduced in the operator algebraic framework by Koelink and Kustermans in [51]. It was further studied together with Groenevelt in [35]. In this reference, a direct integral decomposition of the multiplicative unitary W into irreducible corepresentations is given. The best introduction to $SU_q(1,1)_{\text{ext}}$ is probably to read [35, Sections 3 - 5]. We actually recommend this for a proper understanding of this thesis. However, to keep this thesis self-contained, we recall the results that are necessary for our purposes here.

At this point, it feels appropriate to mention that $SU_q(1,1)_{\text{ext}}$ was obtained by De Commer from a linking quantum groupoid [12]. The main advantage of De Commer's method is that it avoids the lengthy proof of the coassociativity of the coproduct in [51]. Moreover, it is suggested that this gives a possible method for defining operator algebraic deformations of $SU(n,m)$.

Let us define some preliminary notation first.

Definition 1.8.1. For $a \in \mathbb{C}$ and $q \in (0,1)$, we will use the following notation for the *q-shifted factorial*,

$$(a,q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}). \tag{1.15}$$

We let $(a,q)_\infty$ denote the limit $n \to \infty$ of (1.15). Furthermore, we use the common notation

$$(a_0,\ldots,a_1;q)_\infty = (a_0;q)_\infty \cdot \ldots \cdot (a_0;q)_\infty.$$

We need the *θ-product identity*.

Proposition 1.8.2. *For $a \in \mathbb{C}\backslash\{0\}, k \in \mathbb{Z}$, the idenity*

$$(aq^k, q^{1-k}/a; q)_\infty = (-a)^{-k} q^{-\frac{1}{2}k(k-1)} (a, q/a; q)_\infty. \tag{1.16}$$

holds.

Next, we will recall most of the structure of $SU_q(1,1)_{\text{ext}}$. We follow [51] and [35] and keep their notation, using one exception: we will denote $\mathbb{N}_0$ for the natural numbers without 0 and keep denoting the natural numbers by $\mathbb{N}$. Note that $\mathcal{K}$ instead of $\mathcal{H}$ will denote the GNS Hilbert space of $SU_q(1,1)_{\text{ext}}$. The definition below is not complete. We do not really define the comultiplication explicitly but rather say that it is defined by means of (1.1). In fact this is how it is originally defined in [51]. For a more detailed study we refer to the original sources [51], [35].

Definition 1.8.3 (of $SU_q(1,1)_{\text{ext}}$)**.** Let $q \in (0,1)$. We set $I_q = -q^{\mathbb{N}_0} \cup q^{\mathbb{Z}}$ and equip I_q with the counting measure. We define operators α, γ, e, u on the Hilbert space $L^2(\mathbb{T}) \otimes L^2(I_q)$ with standard basis $\zeta^m \otimes \delta_p, m \in \mathbb{Z}, p \in I_q$ as the closed operators determined by:

$$\begin{array}{ll} \alpha(\zeta^m \otimes \delta_p) = \sqrt{\text{sgn}(p) + p^{-2}}\zeta^m \otimes \delta_{qp}, & \gamma(\zeta^m \otimes \delta_p) = p^{-1}\zeta^{m+1} \otimes \delta_p, \\ e(\zeta^m \otimes \delta_p) = \text{sgn}(p)\zeta^m \otimes \delta_p, & u(\zeta^m \otimes \delta_p) = \zeta^m \otimes \delta_{-p}, \end{array}$$

where we interpret δ_{-p} as the zero vector in case $-p \notin I_q$. We let M be the von Neumann algebra generated by α, γ, e and u. It is not difficult to show that $M = L^\infty(\mathbb{T}) \otimes B(L^2(I_q))$.

$SU_q(1,1)_{\text{ext}}$ is unimodular, i.e. the left and right Haar weight are equal. We define this Haar weight as follows. Note that M carries a canonical trace

$$\text{Tr} = \int d\theta \otimes \text{Tr}_{B(L^2(I_q))},$$

where the traces on the right hand side shoud be understood naturally. We set $\mathcal{K} = L^2(\mathbb{T}) \otimes L^2(I_q) \otimes L^2(I_q)$ and denote the canonical basis with $f_{m,p,t}, m \in \mathbb{Z}, p, t \in I_q$. We give a GNS-construction by setting

$$\begin{aligned} \Lambda_{\text{Tr}} : \mathfrak{n}_{\text{Tr}} \to \mathcal{K} :& x \mapsto \sum_{p \in I_q} (x \otimes 1_{L^2(I_q)}) f_{0,p,p}, \\ \pi_{\text{Tr}} : M \to B(\mathcal{K}) :& x \mapsto x \otimes 1_{L^2(I_q)}. \end{aligned}$$

We formally define the weight φ by setting $\varphi(x) = \text{Tr}(|\gamma|x|\gamma|)$. The precise construction is contained in [82] or Appendix A.4. Let us repeat this construction. Let D_0 be the set of all $x \in M$, such that $x|\gamma|$ is closable and $[x|\gamma|] \in \mathfrak{n}_{\text{Tr}}$. One defines a map Λ as the σ-strong-$*$/norm closure of the map

$D_0 \to B(\mathcal{K}) : x \mapsto \Lambda_{\mathrm{Tr}}([x|\gamma|])$. Then, φ is the unique normal, semi-finite, faithful weight such that $(\mathcal{K}, \Lambda, \pi_{\mathrm{Tr}})$ is a GNS-construction for φ. We simply write $(\mathcal{K}, \Lambda, \pi)$ for this GNS-representation. In particular, we get:

$$\varphi(x) = \sum_{p_0 \in I_q} p_0^{-2} \langle \pi(x) f_{0,p_0,p_0}, f_{0,p_0,p_0} \rangle, \qquad x \in M^+. \tag{1.17}$$

The comultiplication is more delicate to define. It is constructed by first constructing the multiplicative unitary and then using relation (1.1). By [51, Discussion below (3.3)], we have:

$$\Delta(e) = e \otimes e.$$

For the action of the comultiplication on the other generators, we refer to [51]. However, note that a simple expression for $\Delta(u)$ is unavailable. The quantum group obtained is called extended quantum $SU(1,1)$ or *the quantum analogue of the normalizer of* $SU(1,1)$ *in* $SL(2,\mathbb{C})$, simply denoted by $SU_q(1,1)_{\mathrm{ext}}$.

Next, we turn to the *dual quantum group* of $SU_q(1,1)_{\mathrm{ext}}$. The existence is guaranteed by the general theory. In addition, it can be studied be means of generators of a universal enveloping (Lie) algebra. The generators are given by operators E, K and two extra generators U_0^{+-}, U_0^{-+} which appear because of the fact that we are looking at an extension of $SU_q(1,1)$. We give explicit formula for E and K and indicate how Groenevelt, Koelink and Kustermans obtain a decomposition of the multiplicative unitary using the Casimir operator Ω.

We start by adding more auxiliary notation to our dictionary, which we borrow from [35].

Notation 1.8.4. Define the following functions:

$$\begin{array}{ll} \mu : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\} : y \mapsto \frac{1}{2}(y + y^{-1}), & \chi : -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} : p \mapsto {}^q\!\log(|p|), \\ \nu : -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} \to \mathbb{R} : t \mapsto q^{\frac{1}{2}(\chi(t)-1)(\chi(t)-2)}, & \kappa : \mathbb{R} \to \mathbb{R} : x \mapsto \mathrm{sgn}(x) x^2, \\ c_q = (\sqrt{2} q (q^2, -q^2; q^2)_\infty)^{-1}. & \end{array}$$

Here, the ${}^q\!\log$ is the ordinary logarithm with base q. We set $v : q^{\mathbb{Z}} \to \mathbb{Z}$ by $v(t) = k$ for $t = q^{2k}$ or $t = q^{2k-1}$.

Definition 1.8.5. We define the operators $E_0, E_0^\dagger, K_0$ which have as domain the finite linear span of vectors $f_{m,p,t}, m \in \mathbb{Z}, p, t \in I_q$ and which are given by

$$\begin{aligned} (q - q^{-1}) E_0 f_{m,p,t} &= \mathrm{sgn}(t) q^{-\frac{m-1}{2}} |p/t|^{\frac{1}{2}} \sqrt{1 + \kappa(q^{-1}t)} f_{m-1,p,q^{-1}t} \\ (q - q^{-1}) E_0^\dagger f_{m,p,t} &= \mathrm{sgn}(t) q^{-\frac{m+1}{2}} |p/t|^{\frac{1}{2}} \sqrt{1 + \kappa(t)} f_{m+1,p,qt} \\ &\quad - \mathrm{sgn}(t) q^{\frac{m-1}{2}} |t/p|^{\frac{1}{2}} \sqrt{1 + \kappa(p)} f_{m-1,qp,t}, \\ K_0 f_{m,p,t} &= q^{-\frac{m}{2}} |p/t|^{\frac{1}{2}} f_{m,p,t}. \end{aligned} \tag{1.18}$$

We let E and K be their closures. We define

$$\Omega_0 = \frac{1}{2}((q-q^{-1})^2 E_0^* E_0 - qK_0^2 - q^{-1}K_0^{-2}).$$

And we define the *Casimir operator* Ω by

$$\Omega = \frac{1}{2}((q-q^{-1})^2 E^* E - qK^2 - q^{-1}K^{-2}). \tag{1.19}$$

Then, Ω is a self-adjoint operator.

At this point, the most natural thing to do is to define the operators U_0^{+-} and U_0^{-+} as in [35]. However, to avoid introducing redundant new terminology, we refer to [35] for its definition for now. After we recalled the direct integral decomosition for W, we give a precise definition of U_0^{+-} and U_0^{-+}.

Definition 1.8.6. The dual von Neumann algebra $\hat{M}$ equals the von Neumann algebra generated by E, K, U_0^{+-} and U_0^{-+}.

In particular, the Casimir operator is affiliated with $\hat{M}$. Moreover, we have the following.

Proposition 1.8.7 (Theorem 4.6 of [35]). *The Casimir operator Ω is the unique self-adjoint extension of Ω_0 that is affiliated with $\hat{M}$.*

The spectral projections of Ω will give invariant subspaces of W. Starting from this observation, Groenevelt et. al. [35] obtain a direct integral decomposition of W. In particular, they write

$$\mathcal{K} = \oplus_{p,m,\varepsilon,\eta} \mathcal{K}(p,m,\varepsilon,\eta), \tag{1.20}$$

where the sum is taken over all $p \in q^{\mathbb{Z}}, m \in \mathbb{Z}, \varepsilon \in \{-1,1\}, \eta \in \{-1,1\}$. The spaces $\mathcal{K}(p,m,\varepsilon,\eta)$ are closed subspaces of $\mathcal{K}$ that are invariant for Ω and moreover, the restriction of Ω to $\mathcal{K}(p,m,\varepsilon,\eta)$ has singular generalized eigenvalues, i.e. [35, Proposition 8.13]. Now [35] proceeds by a careful analysis of the spectral decomposition of Ω restricted to the summands of (1.20). It results in a decomposition of the multiplicative unitary, which is given by the following formula.

Definition 1.8.8. Let W be the multiplicative unitary of (M, Δ). Recall from [51, Proposition 4.5] that it has an explicit description as:

$$\begin{aligned} &W^*(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) \\ = &\sum_{\substack{y,x \in I_q, \\ \mathrm{sgn}(p_2t_2)(yz/p_1)q^{m_2} \in I_q}} |t_2/y| a_{t_2}(p_1,y) a_{p_2}(z, \mathrm{sgn}(p_2t_2)(yz/p_1)q^{m_2}) \\ &\times f_{m_1+m_2-\chi(p_1p_2/t_2z),z,t_1} \otimes f_{\chi(p_1p_2/t_2z),\mathrm{sgn}(p_2t_2)(yz/p_1)q^{m_2},y}. \end{aligned} \tag{1.21}$$

We refer to [51, Definition 3.1] for the definition of the a-function. For many of our purposes only the way W shifts the indices of the basis $f_{m,p,t}$ matters.

The decomposition of W now is of the form

$$W = \bigoplus_{p \in q^{\mathbb{Z}}} \left(\int_{[-1,1]}^{\oplus} W_{p,x} dx \oplus \bigoplus_{x \in \sigma_d(\Omega_p)} W_{p,x} \right). \tag{1.22}$$

Here $\sigma_d(\Omega_p)$ is the discrete spectrum of the Casimir operator [35, Definition 4.5, Theorem 4.6] restricted to the subspace given in [35, Theorem 5.7]. $W_{p,x}$ is a corepresentation that is a direct sum of at most 4 irreducible corepresentations [35, Propositions 5.3 and 5.4], see also below. We simply write $W = \int^{\oplus} W_{p,x} d(p,x)$ for the integral decomposition (1.22). We emphasize that the corepresentations appearing in (1.22) are not mutually inequivalent.

The corepresentations in the continuous part of the decomposition are called *principal series* corepresentations, the corepresentations that appear as a direct summand are called the *discrete series* corepresentations. In addtion the *complementary series* corepresentations $W_{p,x}, x \in (\mu(-q), -1) \cup (1, \mu(q))$, are defined by analytic continuation of matrix coefficients, see [35, Section 10.3]. We mention that it remains unproved that these make up all the corepresentations.

We first focus on the *principal series*. By analytic continuation, the results below also hold for the *complementary part*. So let $p \in q^{\mathbb{Z}}$ and let $x \in (-\mu(q), \mu(q))$ Using the notation of [35, Sections 10.2 - 10.3], an orthonormal basis for the corepresentation Hilbert space $\mathcal{L}_{p,x}$ of the principal and complementary series $W_{p,x}$ is given by the vectors

$$e_m^{\varepsilon,\eta}(p,x), \qquad \varepsilon, \eta \in \{-,+\}, m \in \mathbb{Z}. \tag{1.23}$$

From the considerations in Chapter 4, this notation for the basis is not the most natural one. It is useful to shift this basis. For any $p \in q^{\mathbb{Z}}, x \in (\mu(-q), \mu(q)) \cup \sigma_d(\Omega_p)$, we set

$$f_m^{\varepsilon,\eta}(p,x) = e^{\varepsilon,\eta}_{m-\frac{1}{2}\chi(p)}(p,x), \qquad \varepsilon, \eta \in \{-,+\},$$

where $m \in \mathbb{Z}$ if $p \in q^{2\mathbb{Z}}$ and $m \in \frac{1}{2} + \mathbb{Z}$ if $p \in q^{1+2\mathbb{Z}}$.

Explicitly, the corepresentation $W_{p,x}$ is given by [35, Lemma 10.9 and Section 10.3]. For $p \in q^{\mathbb{Z}}, x \in (\mu(-q), \mu(q))$,

$$\begin{aligned} & \left(\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)}\right) (W_{p,x})\, f_{m_0,p_0,t_0} \\ = \ & C(\varepsilon\eta x; m' - \tfrac{1}{2}\chi(p), \varepsilon', \eta'; \xi\xi'|p_0|q^{-m-m'}, p_0, m-m')\delta_{sgn(p_0),\eta\eta'} \\ & \times\ f_{m_0-m+m', \varepsilon\varepsilon'|p_0|q^{-m-m'}, t_0}. \end{aligned} \tag{1.24}$$

The C-funtion can be given explicitly by means of basic hypergeometric series. For $\lambda \in \mathbb{T}$,

$$\begin{aligned} & C(\mu(\lambda); m, \varepsilon, \eta; p_1, p_2, n) \\ =& \varepsilon^{\frac{1}{2}(1-\mathrm{sgn}(p_1))} \eta^{\frac{1}{2}(1-\mathrm{sgn}(p_2))+n} S(-\mathrm{sgn}(p_1p_2)\lambda; p_1, p_2, n) \frac{A(\lambda; p, m', \varepsilon', \eta')}{A(\mathrm{sgn}(p_1p_2)\lambda; p, m, \varepsilon, \eta)}. \end{aligned} \tag{1.25}$$

Here, the fraction of the A-functions is a fase factor, i.e. a number on the complex unit circle. We will not need an explicit description for them. The S-function has an expression in terms of basic hypergeometric series given by:

$$\begin{aligned}
&S(t;p_1,p_2,n)\\
=&p_2^n q^{\frac{1}{2}n(n-1)}|p_1p_2|\nu(p_1)\nu(p_2)c_q^2\sqrt{(-\kappa(p_1),-\kappa(p_2);q^2)_\infty}\\
\times&\ \frac{(q^2,-q^2/\kappa(p_2),-tq^{3-n}/p_1p_2,-p_1p_2q^{n-1}/t,p_1q^{1-n}/p_2t;q^2)_\infty}{(-p_1|p_2|q^{-n-1}/t,-tq^{n+3}/p_1|p_2|,|p_1|q^{1+n}/|p_2|t;q^2)_\infty}\\
\times&\ (\mathrm{sgn}(p_1p_2)q^{2+2n};q^2)_\infty\,{}_2\varphi_1\left(\begin{matrix}p_2q^{1+n}/p_1t,p_2tq^{1+n}/p_1\\ \mathrm{sgn}(p_1p_2)q^{2+2n}\end{matrix};q^2,-q^2/\kappa(p_2)\right)
\end{aligned}\tag{1.26}$$

The discussion above fully characterizes the principal and complementary series corepresentations of $SU_q(1,1)_{\text{ext}}$.

We will need some extra terminology. We set the vectors:

$$\begin{aligned}
g_m^{1,+}(p,x)=&\frac{1}{2}\sqrt{2}(f_m^{+,+}(p,x)+i^{\chi(p)}f_m^{-,-}(p,x)),\\
g_m^{1,-}(p,x)=&\frac{1}{2}\sqrt{2}(f_m^{+,-}(p,x)-i^{\chi(p)}f_m^{-,+}(p,x)),\\
g_m^{2,+}(p,x)=&\frac{1}{2}\sqrt{2}(f_m^{+,+}(p,x)-i^{\chi(p)}f_m^{-,-}(p,x)),\\
g_m^{2,-}(p,x)=&\frac{1}{2}\sqrt{2}(f_m^{+,-}(p,x)+i^{\chi(p)}f_m^{-,+}(p,x)).
\end{aligned}\tag{1.27}$$

We let

$$\begin{aligned}
\mathcal{L}^1_{p,x}=&\overline{\mathrm{span}}\left\{g_m^{1,+}(p,x),g_m^{1,-}(p,x)\mid m\in\frac{1}{2}\chi(p)+\mathbb{Z}\right\},\\
\mathcal{L}^2_{p,x}=&\overline{\mathrm{span}}\left\{g_m^{2,+}(p,x),g_m^{2,-}(p,x)\mid m\in\frac{1}{2}\chi(p)+\mathbb{Z}\right\}.
\end{aligned}$$

Then, $\mathcal{L}^1_{p,x}$ and $\mathcal{L}^2_{p,x}$ are irreducible subspaces for $W_{p,x}$ if $x\neq 0$. The corresponding corepresentations are denoted by $W^1_{p,x}$ and $W^2_{p,x}$. In case $x=0$, one can prove that these spaces split as a direct sum of two irreducible subspaces. These are denoted by $W^{1,1}_{p,x},W^{1,2}_{p,x},W^{2,1}_{p,x}$ and $W^{2,2}_{p,x}$, see [35] for the exact definition.

Next, we turn to the *discrete series* corepresentations. We copy the following proposition from [35]. It gives a set of basis vectors for the discrete series corepresentations. Instead of giving a formula of these basis vectors in terms of our standard basis $f_{m,p,t},m\in\mathbb{Z},p,t,\in I_q$, we rather characterize the corepresentations $W_{p,x}$ by saying that the same formula (1.24) also holds for the discrete series corepresentations. So the notation of the basis is consistent with the principal series. This then fully characterizes the discrete series corepresentations.

Proposition 1.8.9 (Proposition 5.2 of [35]). *Consider $p \in q^{\mathbb{Z}}$ and $x = \mu(\lambda)$, where $\lambda \in -q^{2\mathbb{Z}+1}p \cup q^{2\mathbb{Z}+1}p$ and $|\lambda| > 1$. Let $j, l \in \mathbb{Z}$ be such that $|\lambda| = q^{1-2j} = q^{1+2l}$, so $l < j$. Then (p, x) determines a discrete series corepresentation of (M, Δ) in the following 3 cases, and these are the only cases:*

1. *If $x > 0$, in which case*

$$\{f_m^{++}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}\} \cup \{f_m^{-+}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}, m \leq l\}$$
$$\cup\{f_m^{+-}(p,x) \mid m \in \frac{1}{2}p + \mathbb{Z}, m \geq j\}$$

   is an orthonormal basis for $\mathcal{L}_{p,x}$.

2. *If $x < 0, l \geq \frac{1}{2}\chi(p)$ and $j > \frac{1}{2}\chi(p)$, in which case*

$$\{f_m^{-+}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}\} \cup \{f_m^{++}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}, m \leq l\}$$
$$\cup\{f_m^{--}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}, m \geq j\}$$

   is an orthonormal basis for $\mathcal{L}_{p,x}$.

3. *If $x < 0, l < \frac{1}{2}\chi(p)$ and $j \leq \frac{1}{2}\chi(p)$, in which case*

$$\{f_m^{+-}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}\} \cup \{f_m^{--}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}, m \leq l\}$$
$$\cup\{f_m^{++}(p,x) \mid m \in \frac{1}{2}\chi(p) + \mathbb{Z}, m \geq j\}$$

   is an orthonormal basis for $\mathcal{L}_{p,x}$.

Let $p \in q^{\mathbb{Z}}$. In particular, the previous proposition implies that $x \in \sigma_d(\Omega_p)$ if and only if $x \in \mu(-q^{2\mathbb{Z}+1}p \cup q^{2\mathbb{Z}+1}p)$. Note that for the discrete series corepresentation $W_{p,x}$ a subset of the vectors (1.23) gives a basis for the corepresentation space $\mathcal{L}_{p,x}$. It is useful to define (1.23) as the zero vector in case $f_m^{\varepsilon,\eta}(p,x)$ is not in one of the sets defined in cases 1 - 3 of Proposition 1.8.9. In particular, the non-zero vectors of (1.23) form an orthonormal basis of $\mathcal{L}_{p,x}$.

Recall that for $x \in [-1, 1]$ the actions of the (unbounded) generators of the dual quantum group, see [35, Eqn. (92)], are given by:

$$\begin{aligned} K f_m^{\varepsilon,\eta}(p,x) &= q^m f_m^{\varepsilon,\eta}(p,x), \\ (q^{-1}-q) E f_m^{\varepsilon,\eta}(p,x) &= q^{-\frac{1}{2}-m}|1+\varepsilon\eta q^{2m+1}e^{i\theta}| f_{m+1}^{\varepsilon,\eta}(p,x), \\ U_0^{+-} f_m^{\varepsilon,\eta}(p,x) &= \eta(-1)^{\upsilon(p)} f_m^{\varepsilon,-\eta}(p,x), \\ U_0^{-+} f_m^{\varepsilon,\eta}(p,x) &= \varepsilon\eta^{\chi(p)}(-1)^{m-\frac{1}{2}\chi(p)} f_m^{-\varepsilon,\eta}(p,x), \end{aligned} \tag{1.28}$$

where θ is such that $x = \mu(e^{i\theta})$. There are similar expression for the generators E, K, U_0^{+-}, U_0^{-+} as in (1.28) for the discrete series corepresentations. They can be found in [35, Lemma 10.1].

Remark 1.8.10. Note that the operator U_0^{+-}, U_0^{-+} can be defined by means of the direct integral of the expressions on the corepresentation spaces.

Remark 1.8.11. For every $p \in q^{\mathbb{Z}}$, $x \in \mu(-q^{2\mathbb{Z}+1}p \cup q^{2\mathbb{Z}+1}p)$, one can define a corepresentation $W_{p,x}$ by defining the action of the generators of $\hat{M}$ by means of [35, Lemma 10.1]. Since the actions of the generators of $W_{pr,x}$ are equal for any $r \in q^{4\mathbb{Z}}$, these corepresentations are all equivalent. Using Proposition 1.8.9, one can check that every such corepresentation is equivalent to at least one corepresentation in the decomposition (1.22). Since every discrete series corepresentation is infinite dimensional, it occurs infinitely many times in the decomposition (1.22) by the Plancherel theorem (of which we prove that it applies), or see Proposition 4.2.4 for a direct proof. Therefore, we see that also:

$$W \simeq \bigoplus_{p \in q^{\mathbb{Z}}} \left(\int_{[-1,1]}^{\oplus} W_{p,x} dx \oplus \bigoplus_{x \in \mu(-q^{2\mathbb{Z}+1}p \cup q^{2\mathbb{Z}+1}p)} W_{p,x} \right). \tag{1.29}$$

Throughout the thesis, we do not make use of this 'second' decomposition of W expect for the definition of the K-functions (4.17), which is more convenient this way.

We end the section by defining a grading on $\hat{M}$ and collecting the last ingredients we need. The grading is of crucial importance in Chapter 4.

Definition 1.8.12. We define closed subspaces $\mathcal{K}_+, \mathcal{K}_- \subseteq \mathcal{K}$ as

$$\mathcal{K}_\pm = \overline{\text{span}}\{f_{m,p,t} \mid m \in \mathbb{Z}, p, t \in I_q \text{ so that } \text{sgn}(pt) = \pm\}.$$

So $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. We define the σ-weakly closed subspaces $\hat{M}_+, \hat{M}_- \subseteq \hat{M}$ as

$$\hat{M}_+ = \{x \in \hat{M} \mid x\mathcal{K}_\pm \subseteq \mathcal{K}_\pm\} \qquad \text{and} \qquad \hat{M}_- = \{x \in \hat{M} \mid x\mathcal{K}_\pm \subseteq \mathcal{K}_\mp\}.$$

Proposition 1.8.13. *$\hat{M} = \hat{M}_+ \oplus \hat{M}_-$. Let $x \in \hat{M}_+$ and $y \in \hat{M}_-$, then $x\Omega \subseteq \Omega x$ and $y\Omega \subseteq -\Omega y$.*

Moreover, it is discussed at the end of Section 3 of [35] how an explicit description of $\hat{J}$ can be found without determining $\hat{\varphi}$. It is given by

$$\hat{J} f_{m,p,t} = \text{sgn}(p)^{\chi(p)} \text{sgn}(t)^{\chi(t)} (-1)^m f_{-m,p,t} \qquad p, t \in I_q, m \in \mathbb{Z}. \tag{1.30}$$

Note that $\hat{J}$ should be extended *anti*-linearly.

Chapter 2

Modular properties of matrix coefficients

In [19] Desmedt obtains a quantum analogue of the Plancherel theorem using so called Duflo-Moore operators. Desmedt's result is based on a paper by Duflo and Moore [24] in which they give a decomposition of the biregular representation of a locally compact group that is generally non-unimodular. The non-unimodularity implies the existence of a field of strictly positive operators, the Duflo-Moore operators, that determine the intertwiner which decomposes the biregular representation. In the quantum setting similar operators play a role even in the unimodular cases.

The main purpose of the present chapter is to investigate what other properties can be related to the Duflo-Moore operators. In particular, we study the question: can the modular properties of a quantum group be interpreted in terms of Duflo-Moore operators?

Here we answer the question in the affirmative. We derive an explicit formula for the modular automorphism group of a unimodular locally compact quantum group in terms of Duflo-Moore operators coming from both the left and right version of the Plancherel theorem. Under sufficient conditions on a quantum group, we obtain the modular formula:

$$\sigma_t \left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U). \tag{2.1}$$

We refer to Theorem 2.2.10 for the exact statement.

As an application, we determine the Duflo-Moore operators for the principle series corepresentations of $SU_q(1,1)_{\mathrm{ext}}$ up to a constant. The constant remains undetermined, since it depends on the choice of the Plancherel-measure. In [19] the Duflo-Moore operators were already determined for the discrete series corepresentations involving summation formulae for basic hypergeometric series. The method we present here is different and applies to the discrete series as well.

We conclude the chapter by computing the undetermined constant for a fixed choice of the Plancherel measure. Here q-analysis gets involved again and is in fact inevitable, since the outcome involves q-special functions. As a corollary this determines the dual Haar weight of $SU_q(1,1)_{\mathrm{ext}}$, which remained absent in the literature so far.

The larger part of the contents of this chapter is included in the joint paper with Erik Koelink [10].

Notation 2.0.1. In Chapter 2 we keep the following notational conventions. (M, Δ) is a locally compact quantum group that satisfies the assumptions of the Plancherel theorem, i.e. $\hat{M}$ is type I and $\hat{M}_u$ is separable. It is also sufficient to assume that $\hat{M}$ is type I and $\hat{M}_c$ is separable. μ denotes a fixed Plancherel measure and D_U and E_U denote the Duflo-Moore operators. Every integral with respect to μ is taken over $\mathrm{IC}(M)$ and we omit the domain most of the times for notational convenience. We use this notation for the direct integrals as well as for the σ-weak integrals we encounter. Similarly, every field of Hilbert spaces, operators, corepresentations, et cetera, is a field over $\mathrm{IC}(M)$ and measurability means μ-measurability. From Section 2.3 on we only consider the quantum group $SU_q(1,1)_{\mathrm{ext}}$ and adopt the notational conventions from Section 1.8.

2.1 Consequences of the Plancherel theorem

This section collects all the consequences of the Plancherel theorem that are necessary for the computations in Sections 2.2 and 2.4. This means that we explicitly state the orthogonality relations that are implicitly contained in the Plancherel theorems. The precise relations can already be found in [19] for the discrete series corepresentations. For compact groups they are explicitly contained in the Peter-Weyl Theorem 1.5.8. However, the more general formulation is unavailable. Proceeding by proving a few lemmas that will be useful later on, we give this formulation. Finally, we are able to prove a theorem that is useful to determine when a field of vectors is in the domain of the Duflo-Moore operators.

Lemma 2.1.1.

1. *Let $x \in M$, such that the linear map $f : \hat{\Lambda}(\lambda(\mathcal{I})) \to \mathbb{C} : \xi(\alpha) \mapsto \alpha(x^*)$ is bounded. Then $x \in \mathfrak{n}_\varphi$ and $f(\xi(\alpha)) = \langle \xi(\alpha), \Lambda(x) \rangle$.*

2. *Let $x \in M$, such that the linear map $f : \hat{\Gamma}(\lambda(\mathcal{I}_R)) \to \mathbb{C} : \xi_R(\alpha) \mapsto \alpha(x^*)$ is bounded. Then $x \in \mathfrak{n}_\psi$ and $f(\xi_R(\alpha)) = \langle \xi_R(\alpha), \Gamma(x) \rangle$.*

Proof. We prove the first statement, the second being analogous. The claim is true for $x \in \mathfrak{n}_\varphi$, since $\mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$ and by definition $\langle \xi(\alpha), \Lambda(x) \rangle = \alpha(x^*)$, for all $\alpha \in \mathcal{I}$. Now, let $x \in M$ be arbitrary. The set $\{\xi(\alpha) \mid \alpha \in \mathcal{I}\}$ is dense in $\mathcal{H}$ by Lemma 1.2.1. Hence, by the Riesz theorem, there is a $v \in \mathcal{H}$ such that for every $\alpha \in \mathcal{I}$, $\alpha(x^*) = \langle \xi(\alpha), v \rangle$. Let $(e_j)_{j \in J}$ be a bounded net in the Tomita algebra $\mathcal{T}_\varphi$ converging σ-weakly to 1 and such that $\sigma_{i/2}(e_j)$ converges σ-weakly to 1, see Lemma A.6.2. Let $a, b \in \mathcal{T}_\varphi$ and fix the normal functional α by $\alpha(x) = \varphi(axb), x \in M$. Using Lemma 1.2.1 again, we find

$$
\begin{aligned}
&\langle \xi(\alpha), \Lambda(xe_j) \rangle = \varphi(ae_j^* x^* b) \\
=&\langle \Lambda(b\sigma_{-i}(ae_j^*)), v \rangle = \langle \Lambda(b\sigma_{-i}(a)), J\pi(\sigma_{i/2}(e_j)^*)Jv \rangle.
\end{aligned}
$$

Hence, $\Lambda(xe_j) = J\pi(\sigma_{i/2}(e_j)^*)Jv$, so that $\Lambda(xe_j)$ converges weakly to v. Since $xe_j \to x$ σ-weakly and Λ is σ-weak/weakly closed, this implies that $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$ and $v = \Lambda(x)$. □

Recall that $\int^\oplus B_2(\mathcal{H}_U) d\mu(U) \simeq \int^\oplus \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U)$. For $\eta = \int^\oplus \eta_U d\mu(U), \xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$ the measurable field of vectors $(\xi_U \otimes \overline{\eta_U})_U$ is not necessarily square integrable. If it is square integrable, $\int^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U) \in \int^\oplus B_2(\mathcal{H}_U) d\mu(U)$.

We obtain the following expression for the left Plancherel transform.

Lemma 2.1.2. *Let $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$ and $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$ be such that $\eta \in \mathrm{Dom}(D^{-1})$ and $(\xi_U \otimes \overline{\eta_U})_U$ is square integrable. Then, $\mathrm{IC}(M) \ni U \mapsto (\iota \otimes \omega_{\xi_U, D_U^{-1}\eta_U})(U^*) \in M$ is σ-weakly integrable with respect to μ, we have $\int (\iota \otimes \omega_{\xi_U, D_U^{-1}\eta_U})(U^*) d\mu(U) \in \mathfrak{n}_\varphi$, and*

$$
\mathcal{Q}_L^{-1}\Big(\int^\oplus_{\mathrm{IC}(M)} \xi_U \otimes \overline{\eta_U} d\mu(U)\Big) = \Lambda\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, D_U^{-1}\eta_U})(U^*) d\mu(U)\right). \quad (2.2)
$$

Proof. For $\alpha \in \mathcal{I}$, Theorem 1.6.1 implies that

$$
\begin{aligned}
&\langle \xi(\alpha), \mathcal{Q}_L^{-1}\Big(\int^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U)\Big) \rangle = \int \langle (\alpha \otimes \iota)(U) D_U^{-1}, \xi_U \otimes \overline{\eta_U} \rangle_{\mathrm{HS}} d\mu(U) \\
=& \int (\alpha \otimes \omega_{D_U^{-1}\eta_U, \xi_U})(U) d\mu(U) = \alpha\Big(\int (\iota \otimes \omega_{\xi_U, D_U^{-1}\eta_U})(U^*) d\mu(U)^*\Big),
\end{aligned}
$$

where the last integral exists in the σ-weak sense. Here, we used the ad hoc notation $\langle \cdot, \cdot \rangle_{\mathrm{HS}}$ for the Hilbert-Schmidt inner product. We see by Lemma 2.1.1 that, $\int (\iota \otimes \omega_{\xi_U, D_U^{-1}\eta_U})(U^*) d\mu(U) \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$, and (2.2) follows. □

Remark 2.1.3. As in the proof of Lemma 2.1.2 we see that for $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$, $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$, the σ-weak integral $\int (\iota \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U) \in M$ exists, and for $\alpha \in M_*$, $|\int (\alpha \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U)| \leq \|\alpha\| \|\xi\| \|\eta\|$.

The previous lemma shows that $\mathcal{Q}_L^{-1}$ is an analogue of what Desmedt calls the (left) Wigner map [19, Section 3.3.1]. This map is defined as

$$B_2(\mathcal{H}_U) \to \mathcal{H} : \xi \otimes \overline{\eta} \mapsto \Lambda\left((\iota \otimes \omega_{\xi, D_U^{-1}\eta})(U^*)\right), \tag{2.3}$$

where U is a corepresentation on a Hilbert space $\mathcal{H}_U$ that appears as a discrete mass point in the Plancherel measure, cf. Remark 1.6.5. This map is also considered in [5, p. 203], where it is denoted by Φ.

The next lemma is the right analogue of Lemma 2.1.2, the proof being similar.

Lemma 2.1.4. *Let $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$ and $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$ be such that $\eta \in \mathrm{Dom}(E^{-1})$ and $(\xi_U \otimes \overline{\eta_U})_U$ is square integrable. Then $\mathrm{IC}(M) \ni U \mapsto (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) \in M$ is σ-weakly integrable with respect to μ. Furthermore, $\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) d\mu(U) \in \mathfrak{n}_\psi$ and*

$$\mathcal{Q}_R^{-1}\left(\int_{\mathrm{IC}(M)}^\oplus \xi_U \otimes \overline{\eta_U} d\mu(U)\right) = \Gamma\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) d\mu(U)\right).$$

Now, we are able to give orthogonality relations between integrals of matrix coefficients. The orthogonality relations will be used in Section 2.4 where we give a method to determine the Duflo-Moore operators of a locally compact quantum group that satisfies the assumptions of the Plancherel theorem.

Theorem 2.1.5 (Orthogonality relations)**.** *Let (M, Δ) be a locally compact quantum group, such that $\hat{M}_u$ is separable and $\hat{M}$ is a type I von Neumann algebra. Let $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$, $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$, $\eta' = \int^\oplus \eta'_U d\mu(U) \in \mathcal{H}$ and $\xi' = \int^\oplus \xi'_U d\mu(U) \in \mathcal{H}$. We have the following orthogonality relations:*

1. *Suppose that $\eta, \eta' \in \mathrm{Dom}(D)$ and that $(\xi_U \otimes \overline{D_U \eta_U})_U, (\xi'_U \otimes \overline{D_U \eta'_U})_U$ are square integrable fields of vectors, then*

$$\varphi\left(\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U^*) d\mu(U)\right)^* \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi'_U, \eta'_U})(U^*) d\mu(U)\right) = \int_{\mathrm{IC}(M)} \langle D_U \eta_U, D_U \eta'_U\rangle \langle \xi'_U, \xi_U\rangle d\mu(U). \tag{2.4}$$

2. *Suppose that $\eta, \eta' \in \mathrm{Dom}(E)$ and that $(\xi_U \otimes \overline{E_U \eta_U})_U, (\xi'_U \otimes \overline{E_U \eta'_U})_U$ are square integrable fields of vectors, then:*

$$\psi\left(\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U)\right)^* \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi'_U, \eta'_U})(U) d\mu(U)\right) = \int_{\mathrm{IC}(M)} \langle E_U \eta_U, E_U \eta'_U\rangle \langle \xi'_U, \xi_U\rangle d\mu(U). \tag{2.5}$$

Here $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)$, $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)$, $\int(\iota\otimes\omega_{\xi'_U,\eta'_U})(U)d\mu(U)$, $\int(\iota\otimes\omega_{\xi'_U,\eta'_U})(U^*)d\mu(U)$ *are defined in Lemma 2.1.2 and 2.1.4. The integrals are taken over* $\mathrm{IC}(M)$.

As was mentioned in Remark 1.6.4 the separability assumption on $\hat{M}_u$ can be replaced by separability of $\hat{M}_c$ in Theorem 2.1.5.

As observed in Remark 2.1.3 the element $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)\in M$ exists for $\eta=\int^{\oplus}\eta_U d\mu(U)\in\mathcal{H}$, $\xi=\int^{\oplus}\xi_U d\mu(U)\in\mathcal{H}$ and the next theorem investigates the consequences of $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)\in\mathfrak{n}_\varphi$. Again, this is a useful tool for determining Duflo-Moore operators, as in Section 2.4.

Theorem 2.1.6. *Let* (M,Δ) *be a locally compact quantum group, such that* $\hat{M}_u$ *is separable and* $\hat{M}$ *is a type I von Neumann algebra. Let* $\xi=\int^{\oplus}\xi_U d\mu(U)\in\mathcal{H}$ *be an essentially bounded field of vectors.*

1. *Let* $\eta=\int^{\oplus}\eta_U d\mu(U)\in\mathcal{H}$ *be such that* $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)\in\mathfrak{n}_\varphi$. *Then, for almost every* U *in the support of* $(\xi_U)_U$, *we have* $\eta_U\in\mathrm{Dom}(D_U)$.

2. *Let* $\eta=\int^{\oplus}\eta_U d\mu(U)\in\mathcal{H}$ *be such that* $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)\in\mathfrak{n}_\psi$. *Then, for almost every* U *in the support of* $(\xi_U)_U$, *we have* $\eta_U\in\mathrm{Dom}(E_U)$.

Proof. We only give a proof of the first statement. Consider the sesquilinear form

$$q(\eta,\eta')=\varphi\left(\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)^*\int(\iota\otimes\omega_{\xi_U,\eta'_U})(U^*)d\mu(U)\right),$$

with $q(\eta)=q(\eta,\eta)$,

$$\mathrm{Dom}(q)=\left\{\eta=\int^{\oplus}\eta_U d\mu(U)\mid\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)\in\mathfrak{n}_\varphi\right\}.$$

q is a closed form on $\mathcal{H}$. Indeed, assume that $\eta_n\in\mathrm{Dom}(q)$ converges in norm to $\eta\in\mathcal{H}$ and that $q(\eta_n-\eta_m)\to 0$. Then $\int(\iota\otimes\omega_{\xi_U,\eta_{n,U}})(U^*)d\mu(U)$ converges to $\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)$ σ-weakly. By assumption $\Lambda(\int(\iota\otimes\omega_{\xi_U,\eta_{n,U}})(U^*)d\mu(U))$ is a Cauchy sequence in norm. The σ-weak/weak closedness of Λ implies that

$$\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U)\in\mathrm{Dom}(\Lambda)=\mathfrak{n}_\varphi,$$

so $\eta\in\mathrm{Dom}(q)$ and

$$\Lambda(\int(\iota\otimes\omega_{\xi_U,\eta_{n,U}})(U^*)d\mu(U))\to\Lambda(\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U))$$

weakly. Since we know that $\Lambda(\int(\iota\otimes\omega_{\xi_U,\eta_{n,U}})(U^*)d\mu(U))$ is actually a Cauchy sequence in the norm topology, it is norm convergent to $\Lambda(\int(\iota\otimes\omega_{\xi_U,\eta_U})(U^*)d\mu(U))$. This proves that $q(\eta-\eta_n)\to 0$.

Since $(\xi_U)_U$ is a square integrable, essentially bounded field of vectors, $\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U) \in B_2(\mathcal{H})$. By Lemma 2.1.2, $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(q)$, so that q is densely defined. q is symmetric and positive by its definition. By [47, Theorem VI.2.23], there is a unique positive, self-adjoint, possibly unbounded operator A on $\mathcal{H}$ such that $q(\eta, \eta') = \langle A\eta, A\eta' \rangle$ and $\mathrm{Dom}(A) = \mathrm{Dom}(q)$. By Theorem 2.1.5 we see that for $\eta, \eta' \in \mathrm{Dom}(D)$ we have $\int \langle D_U \eta_U, D_U \eta'_U \rangle \|\xi_U\|^2 d\mu(U) = \langle A\eta, A\eta' \rangle$. Since both A and the multiplication operator $\int^{\oplus} \|\xi_U\| d\mu(U)$ are positive, self-adjoint operators this yields $A = \int^{\oplus} \|\xi_U\| D_U d\mu(U)$. In particular $\eta_U \in \mathrm{Dom}(D_U)$ for almost every $U \in \mathrm{supp}\,((\xi_U)_U) = \overline{\{U \in \mathrm{IC}(M) \mid \|\xi_U\| \neq 0\}}$. $\square$

2.2 Modular formula

In this section we prove the formula for the modular automorphism group as announced in the introduction of this chapter. The idea of proving this formula is to describe the polar decomposition of the closure of the map

$$\Gamma(x) \mapsto \Lambda(x^*), \qquad x \in \mathfrak{n}_\psi \cap \mathfrak{n}_\varphi^* \tag{2.6}$$

explicitly in terms of corepresentations. So in praticular, we give a description of Tomita-Takesaki theory in terms of corepresentations. Then, for a unimodular quantum group, where $\Gamma = \Lambda$, the modular automorphism group is implemented by the absolute value of this operator. Note that this operator is one of the main tools in Connes' proof of his cocycle derivative theorem [75, Section VIII.3].

Let T' be the closure of the linear map (2.6). Recall that $\mathcal{Q}_L$ and $\mathcal{Q}_R$ denote the Plancherel transforms. We describe the polar decomposition of $\mathcal{Q}_L T' \mathcal{Q}_R^{-1}$. Then, we conjugate with the Plancherel transforms $\mathcal{Q}_L$ and $\mathcal{Q}_R$ to find the polar decomposition of T'. Eventually, this yields Theorems 2.2.8 and 2.2.9.

We start with the following technical lemma.

Lemma 2.2.1. *Consider the standard measure space $(\mathrm{IC}(M), \mu)$. Let $(\mathcal{K}_U)_U$ and $(\mathcal{L}_U)_U$ be any measurable fields of Hilbert spaces. Let $(A_U)_U$ and $(B_U)_U$ be measurable fields of possibly unbounded, closed operators on $(\mathcal{K}_U)_U$ and $(\mathcal{L}_U)_U$ respectively. Let $(e_U^n)_U, n \in \mathbb{N}$, be a fundamental sequence for $(A_U)_U$ and let $(f_U^n)_U, n \in \mathbb{N}$ be a fundamental sequence for $(B_U)_U$. Set $A = \int^{\oplus} A_U d\mu(U)$, $B = \int^{\oplus} B_U d\mu(U)$, $\mathcal{K} = \int^{\oplus} \mathcal{K}_U d\mu(U)$ and $\mathcal{L} = \int^{\oplus} \mathcal{L}_U d\mu(U)$.*

(a) $(A_U \otimes B_U)_U$ is a measurable field of closed operators.

(b) The countable set

$$I = \{(e_U^n \otimes f_U^m)_U \mid n, m \in \mathbb{N}\},$$

is a fundamental sequence for $(A_U \otimes B_U)_U$.

(c) The set

$$H = \mathrm{span}\left\{ \int^{\oplus} \xi_U \otimes \eta_U d\mu(U) \mid \textit{Property } (T) \textit{ holds} \right\},$$

$$(T) = \begin{cases} \xi = \int^{\oplus} \xi_U d\mu(U) \in \mathrm{Dom}(A), \\ \eta = \int^{\oplus} \eta_U d\mu(U) \in \mathrm{Dom}(B), \\ \int^{\oplus} (\xi_U \otimes \eta_U) d\mu(U) \in \mathrm{Dom}(\int^{\oplus} (A_U \otimes B_U) d\mu(U)). \end{cases}$$

is a core for $\int^{\oplus} (A_U \otimes B_U) d\mu(U)$.

Remark 2.2.2. There is nothing special about the choice for the standard measure space $(\mathrm{IC}(M), \mu)$. The lemma holds for an arbitrary standard measure space (X, ν). Any field, integral or measurability assumption should be understood with respect to ν then. Here we only need the case $X = \mathrm{IC}(M)$ and $\nu = \mu$, the Plancherel measure. Typically $\mathcal{K}_U$ will be the corepresentation space $\mathcal{H}_U$ of U and $\mathcal{L}_U$ will be the conjugate space $\overline{\mathcal{H}}_U$. This formulation was chosen for notational convenience.

Proof. We first prove (a) and (b). By [21, II.1.8, Proposition 10], for $(\xi_U)_U$, $(\eta_U)_U$ measurable fields of vectors, there is a unique measurable structure so that $(\xi_U \otimes \eta_U)_U$ is a measurable field of vectors. We check (1) - (3) of Definition A.1.11, which is [61, Remark 1.5, (1) - (3)].

(1) $(e_U^n \otimes f_U^m)_U$ is a measurable field of vectors and $e_U^n \otimes f_U^m \in \mathrm{Dom}(A_U \otimes B_U)$ for all U. The function

$$U \mapsto \langle (A_U \otimes B_U)(e_U^n \otimes f_U^m)_U, (e_U^{n'} \otimes f_U^{m'})_U \rangle = \langle A_U e_U^n, e_U^{n'} \rangle \langle B_U f_U^m, f_U^{m'} \rangle,$$

is measurable, so (2) follows. For (3) fix a $U \in \mathrm{IC}(M)$. By definition $\{e_U^n \mid n \in \mathbb{N}\}$ is a core for A_U and $\{f_U^n \mid n \in \mathbb{N}\}$ is a core for B_U. Then, it follows from [42, Lemma 11.2.29] that $\mathrm{span}\{e_U^n \otimes f_U^m \mid n, m \in \mathbb{N}\}$ is a core for $A_U \otimes B_U$, so that I is total in $\mathrm{Dom}(A_U \otimes B_U)$ with respect to the graph norm. In all, we have proved (a) and (b).

Using [21, II.1.3, Remarque 1], we may assume that $(e_U^n)_U$ (resp. $(f_U^n)_U$) satisfies $U \mapsto \|(e_U^n)_U\|$ (resp. $U \mapsto \|(f_U^n)_U\|$) is bounded and vanishes outside a set of finite measure. Let

$$\lambda_U^{n,m} = \left(\max(1, \|(A_U \otimes B_U)(e_U^n \otimes f_U^m)\|, \|A_U e_U^n\|, \|B_U f_U^m\|)\right)^{-1},$$

so $\lambda_U^{n,m}$ is measurable and $0 < \lambda_U^{n,m} \leq 1$. Using the assumption, we find $\int^{\oplus} \lambda_U^{n,m} (e_U^n \otimes f_U^m) d\mu(U) \in H$, i.e. (T) holds. Moreover, $U \mapsto \|\lambda_U^{n,m}(e_U^n \otimes f_U^m)\|^2_{\mathrm{Graph}(A_U \otimes B_U)}$ is bounded. Let $F = \{\int^{\oplus} \lambda_U^{n,m}(e_U^n \otimes f_U^m) d\mu(U) \mid n, m \in \mathbb{N}\} \subseteq H$. Now define

$$G = \bigcup_{f \in \mathcal{C}} m_f F,$$

where $\mathcal{C}$ is the set of bounded measurable scalar-valued functions vanishing outside a set of finite measure and m_f is multiplication by f. Then $G \subseteq H \subseteq$

$\mathrm{Dom}(\int^{\oplus}(A_U \otimes B_U)d\mu(U))$ and by [21, II.1.6, Proposition 7], G is total in the Hilbert space $\mathrm{Dom}(\int^{\oplus}(A_U \otimes B_U)d\mu(U))$ equipped with the graph norm. Hence, H is a core for $\int^{\oplus}(A_U \otimes B_U)d\mu(U)$. □

Notation 2.2.3. Let $\xi = \int^{\oplus} \xi_U d\mu(U) \in \mathcal{H}$ and $\eta = \int^{\oplus} \eta_U d\mu(U) \in \mathcal{H}$, and $A = \int^{\oplus} A_U d\mu(U)$, $B = \int^{\oplus} B_U d\mu(U)$ decomposable operators on $\mathcal{H}$. We will use the notation,

$$(\xi, \overline{\eta}) \in \mathcal{D}^{\otimes}(A, \overline{B}),$$

to indicate the following:

(1) $\xi \in \mathrm{Dom}(A)$;
(2) $\overline{\eta} \in \mathrm{Dom}(\overline{B})$;
(3) $(\xi_U \otimes \overline{\eta_U})_U$ is square integrable and $\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U) \in \mathrm{Dom}(\int^{\oplus}(A_U \otimes \overline{B_U})d\mu(U))$.

For closed operators A and B the set of $\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U)$ with $(\xi_U, \overline{\eta_U}) \in \mathcal{D}^{\otimes}(A, \overline{B})$ is a core for $\int^{\oplus}(A_U \otimes \overline{B_U})d\mu(U)$ by Lemma 2.2.1. In particular this set is dense in $\int^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U)$.

Lemma 2.2.4. *For* $\eta = \int^{\oplus} \eta_U d\mu(U), \xi = \int^{\oplus} \xi_U d\mu(U) \in \mathcal{H}$*, with* $(\eta, \overline{\xi}) \in \mathcal{D}^{\otimes}(E^{-1}, \overline{D})$*, we have* $\mathcal{Q}_R^{-1}\left(\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U)\right) \in \mathrm{Dom}(T')$ *and:*

$$\mathcal{Q}_L T' \mathcal{Q}_R^{-1}\left(\int^{\oplus}_{\mathrm{IC}(M)} \xi_U \otimes \overline{\eta_U} d\mu(U)\right) = \left(\int^{\oplus}_{\mathrm{IC}(M)} E_U^{-1}\eta_U \otimes \overline{D_U \xi_U} d\mu(U)\right). \quad (2.7)$$

Proof. By Lemma 2.1.4,

$$\mathcal{Q}_R^{-1}\left(\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U)\right) = \Gamma\left(\int (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) d\mu(U)\right). \quad (2.8)$$

By Lemmas 2.1.2 and 2.1.4 we obtain,

$$\int (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) d\mu(U) = \left(\int (\iota \otimes \omega_{E_U^{-1}\eta_U, \xi_U})(U^*) d\mu(U)\right)^* \in \mathfrak{n}_\psi \cap \mathfrak{n}_\varphi^*.$$

Hence, by (2.6), (2.8) and Lemma 2.1.2

$$\begin{aligned}\mathcal{Q}_L T' \mathcal{Q}_R^{-1}\left(\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U)\right) &= \mathcal{Q}_L\left(\Lambda\left(\int (\iota \otimes \omega_{\xi_U, E_U^{-1}\eta_U})(U) d\mu(U)^*\right)\right) \\ &= \left(\int^{\oplus} E_U^{-1}\eta_U \otimes \overline{D_U \xi_U} d\mu(U)\right),\end{aligned}$$

from which the lemma follows. □

Let Σ be the anti-linear flip:

$$\Sigma : \int^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U) \to \int^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U)$$
$$\int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U) \mapsto \int^{\oplus} \eta_U \otimes \overline{\xi_U} d\mu(U).$$

Note that Σ is an anti-linear isometry. We are now able to give the polar decomposition of $\mathcal{Q}_L T' \mathcal{Q}_R^{-1}$.

Proposition 2.2.5. *Consider $T'_{\mathcal{Q}} := \mathcal{Q}_L T' \mathcal{Q}_R^{-1}$ as an operator on $\int^{\oplus}_{\mathrm{IC}(M)} \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U)$. Then the polar decomposition of $T'_{\mathcal{Q}}$ is given by*

$$T'_{\mathcal{Q}} = \Sigma \int^{\oplus}_{\mathrm{IC}(M)} D_U \otimes \overline{E_U^{-1}} d\mu(U).$$

Proof. Throughout the proof, let $\eta = \int^{\oplus} \eta_U d\mu(U) \in \mathcal{H}$, $\xi = \int^{\oplus} \xi_U d\mu(U) \in \mathcal{H}$, $\eta' = \int^{\oplus} \eta'_U d\mu(U) \in \mathcal{H}$ and $\xi' = \int^{\oplus} \xi'_U d\mu(U) \in \mathcal{H}$ be such that $(\eta_U \otimes \overline{\xi_U})_U$ and $(\eta'_U \otimes \overline{\xi'_U})_U$ are square integrable.

Assume $(\eta, \overline{\xi}) \in \mathcal{D}^{\otimes}(D, \overline{E^{-1}})$, $(\xi', \overline{\eta'}) \in \mathcal{D}^{\otimes}(D, \overline{E^{-1}})$, so that by (2.7),

$$\begin{aligned}
&\langle \int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U), T'_{\mathcal{Q}} \int^{\oplus} (\xi'_U \otimes \overline{\eta'_U}) d\mu(U) \rangle \\
=&\langle \int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U), \int^{\oplus} (E_U^{-1} \eta'_U \otimes \overline{D_U \xi'_U}) d\mu(U) \rangle \\
=& \int \langle \xi_U, E_U^{-1} \eta'_U \rangle \langle D_U \xi'_U, \eta_U \rangle d\mu(U) \\
=& \int \langle E_U^{-1} \xi_U, \eta'_U \rangle \langle \xi'_U, D_U \eta_U \rangle d\mu(U) \\
=&\langle \int^{\oplus} (\xi'_U \otimes \overline{\eta'_U}) d\mu(U), \int^{\oplus} (D_U \eta_U \otimes \overline{E_U^{-1} \xi_U}) d\mu(U) \rangle.
\end{aligned}$$

So $T'^{*}_{\mathcal{Q}} \left(\int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U) \right) = \int^{\oplus} (D_U \eta_U \otimes \overline{E_U^{-1} \xi_U}) d\mu(U)$.

Assuming $(\xi, \overline{\eta}) \in \mathcal{D}^{\otimes}(D^2, \overline{E^{-2}})$, it follows that

$$T'^{*}_{\mathcal{Q}} T'_{\mathcal{Q}} \left(\int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U) \right) = \int^{\oplus} (D_U^2 \xi_U \otimes \overline{E_U^{-2} \eta_U}) d\mu(U).$$

$\int^{\oplus} D_U^2 \otimes \overline{E_U^{-2}} d\mu(U)$ is a positive, self-adjoint operator for which the set

$$H = \operatorname{span} \left\{ \int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U) \mid (\xi, \eta) \in \mathcal{D}^{\otimes}(D^2, E^{-2}) \right\},$$

forms a core by Lemma 2.2.1. Since $T_{\mathcal{Q}}'^* T_{\mathcal{Q}}'$ is self-adjoint and agrees with the self-adjoint operator $\int^{\oplus} D_U^2 \otimes \overline{E_U^{-2}} d\mu(U)$ on H we find $T_{\mathcal{Q}}'^* T_{\mathcal{Q}}' = \int^{\oplus} D_U^2 \otimes \overline{E_U^{-2}} d\mu(U)$.

Assuming that $(\xi, \overline{\eta}) \in \mathcal{D}^{\otimes}(D, \overline{E^{-1}})$,

$$\Sigma \int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U) \left(\int^{\oplus} (\xi_U \otimes \overline{\eta_U}) d\mu(U) \right) =$$

$$\Sigma \left(\int^{\oplus} (D_U \xi_U \otimes \overline{E_U^{-1} \eta_U}) d\mu(U) \right) = \int^{\oplus} (E_U^{-1} \eta_U \otimes \overline{D_U \xi_U}) d\mu(U),$$

so that $T_{\mathcal{Q}}'$ and $\Sigma \left(\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U) \right)$ agree on a core, cf. Notation 2.2.3. □

Finally we translate everything back to the level of the GNS-space $\mathcal{H}$.

Proposition 2.2.6. *Let*

$$D_{\nabla_0'^{\frac{1}{2}}} = \mathrm{span}\{ \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \mid \eta \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E^{-1}), (\xi, E\eta) \in \mathcal{D}^{\otimes}(D, E^{-1}) \},$$

and define $\nabla_0'^{\frac{1}{2}} : \Gamma(D_{\nabla_0'^{\frac{1}{2}}}) \to \mathcal{H}$ *by*

$$\Gamma\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U)\right) \mapsto \Gamma\left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U \xi_U, E_U^{-1} \eta_U})(U) d\mu(U)\right).$$

Then $\nabla_0'^{\frac{1}{2}}$ *is a densely defined, preclosed operator and its closure* $\nabla'^{\frac{1}{2}}$*, is a self-adjoint, strictly positive operator satisfying* $\mathcal{Q}_R \nabla'^{\frac{1}{2}} \mathcal{Q}_R^{-1} = \int_{\mathrm{IC}(M)}^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)$.

Proof. Set

$$H = \mathrm{span} \left\{ \int^{\oplus} \xi_U \otimes \overline{\eta_U} d\mu(U) \mid (\xi, \eta) \in \mathcal{D}^{\otimes}(D_U^2, E_U^{-2}) \right\}.$$

Then, H is a core for $\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)$. Indeed, H is a core for $\int^{\oplus} D_U^2 \otimes \overline{E_U^{-2}} d\mu(U)$ by Lemma 2.2.1, and hence this is a core for $\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)$.

Now, let $\eta = \int^{\oplus} \eta_U d\mu(U) \in H$ and $\xi = \int^{\oplus} \xi_U d\mu(U) \in H$ be such that

$$\eta \in \mathrm{Dom}(E) \cap \mathrm{Dom}(E^{-1}), \qquad (\xi, E\eta) \in \mathcal{D}^{\otimes}(D, E^{-1}).$$

So $\eta \in \mathrm{Dom}(E)$ and $(\xi_U \otimes E_U \eta_U)_U$ is square integrable, so that

$$\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \in \mathfrak{n}_{\psi},$$

by Lemma 2.1.4. Similarly, since $E^{-1}\eta \in \mathrm{Dom}(E)$ and $(D_U\xi_U \otimes \eta_U)_U$ is square integrable, $\int(\iota \otimes \omega_{D_U\xi_U, E_U^{-1}\eta_U})(U)d\mu(U)) \in \mathfrak{n}_\psi$. Furthermore, we have the following inclusions:

$$H \subseteq \mathcal{Q}_R(\Gamma(D_{\nabla_0'^{\frac{1}{2}}})) \subseteq \mathrm{Dom}(\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)),$$

and for $x \in D_{\nabla_0'^{\frac{1}{2}}}$ we have,

$$\nabla_0'^{\frac{1}{2}}\Gamma(x) = \mathcal{Q}_R^{-1}\left(\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)\right)\mathcal{Q}_R\Gamma(x).$$

Since $\mathcal{Q}_R$ is an isometric isomorphism, the claims follow from the fact that $\int^{\oplus} D_U \otimes \overline{E_U^{-1}} d\mu(U)$ is a self-adjoint, strictly positive operator for which H is a core. □

Proposition 2.2.7. *Let*

$$D_{J_0'} = \mathrm{span}\{\int^{\oplus}_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U) \mid$$
$$\xi \in \mathit{Dom}(D^{-1}), \eta \in \mathit{Dom}(E), (\xi_U \otimes \overline{E_U\eta_U})_U \text{ is square integrable}\},$$

and define $J_0' : \Gamma(D_{J_0'}) \to \mathcal{H}$:

$$\Gamma(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U)) \mapsto \Lambda(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U^{-1}\xi_U, E_U\eta_U})(U)d\mu(U)^*).$$

Then J_0' is a densely defined anti-linear isometry, and its closure, denoted by J', is a surjective anti-linear isometry satisfying $\mathcal{Q}_L J' \mathcal{Q}_R^{-1} = \Sigma$.

Proof. Let

$$H = \mathrm{span}\left\{\int \xi_U \otimes \overline{\eta_U} d\mu(U) \mid (\xi, \eta) \in \mathcal{D}^{\otimes}(D^{-1}, \overline{E})\right\}.$$

Then, H is dense in $\int^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U} d\mu(U)$, c.f. Notation 2.2.3.

For $\eta = \int^{\oplus} \eta_U d\mu(U) \in \mathcal{H}$ and $\xi = \int^{\oplus} \xi_U d\mu(U) \in \mathcal{H}$ so that $\xi \in \mathrm{Dom}(D^{-1})$, $\eta \in \mathrm{Dom}(E)$ and $(\xi_U \otimes \overline{E_U\eta_U})_U$ is square integrable, we find

$$\int (\iota \otimes \omega_{\xi_U, \eta_U})(U)d\mu(U) \in \mathfrak{n}_\psi, \text{ and } \int (\iota \otimes \omega_{D_U^{-1}\xi_U, E_U\eta_U})(U)d\mu(U)^* \in \mathfrak{n}_\varphi$$

by Lemmas 2.1.2 and 2.1.4. So $H \subseteq \mathcal{Q}_R(\Gamma(D_{J_0'}))$. Moreover, for $x \in D_{J_0'}$, $J_0\Gamma(x) = \mathcal{Q}_L^{-1}\Sigma\mathcal{Q}_R\Gamma(x)$. Then, since $\mathcal{Q}_L$ and $\mathcal{Q}_R$ are isomorphisms, the claim follows from the fact that Σ is a surjective anti-linear isometry. □

Note that Proposition 2.2.7 is an analogy of the classical situation. Suppose that G is a locally compact group for which the classical Plancherel theorem [22, Theorem 18.8.1] holds. The anti-linear operator $f \mapsto f^*$ acting on $L^2(G)$ is transformed into the anti-linear flip acting on $\int_{\mathrm{IR}(G)}^{\oplus} \mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi d\mu(\pi)$ by the Plancherel transform. Here $f^*(x) = \overline{f(x^{-1})}\delta_G(x^{-1})$ and δ_G is the modular function on G.

From Proposition 2.2.5 and Propositions 2.2.6 and 2.2.7 we obtain the following result.

Theorem 2.2.8. *The polar decomposition of T' is given by $T' = J'\nabla'^{\frac{1}{2}}$.*

The roles of φ and ψ can be interchanged. Let T'' be the closure of the linear map

$$\mathcal{H} \to \mathcal{H} : \Lambda(x) \mapsto \Gamma(x^*), \quad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi^*. \tag{2.9}$$

The polar decomposition of T'' can be expressed in terms of corepresentations in a similar way.

Theorem 2.2.9. *Consider $T'' : \mathcal{H} \to \mathcal{H}$. Let*

$$D_{J_0''} = \operatorname{span}\{\int_{\mathrm{IC}(M)}^{\oplus} (\iota \otimes \omega_{\xi_U,\eta_U})(U)^* d\mu(U) \mid \\ \xi \in \operatorname{Dom}(D), \eta \in \operatorname{Dom}(E^{-1}), (D_U\xi_U \otimes \overline{\eta_U})_U \text{ is square integrable}\},$$

and define $J_0'' : \Lambda(D_{J_0''}) \to \mathcal{H}$:

$$\Lambda(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U,\eta_U})(U)^* d\mu(U)) \mapsto \Gamma(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U\xi_U, E_U^{-1}\eta_U})(U) d\mu(U)).$$

Then, J_0'' is densely defined and isometric, and its closure, denoted by J'', is a surjective anti-linear isometry. Let

$$D_{\nabla''^{\frac{1}{2}}_0} = \operatorname{span}\{\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U) \mid \text{where} \\ \xi \in \operatorname{Dom}(D) \cap \operatorname{Dom}(D^{-1}), (D\xi, \overline{\eta}) \in \mathcal{D}^{\otimes}(D^{-1}, \overline{E})\},$$

and define $\nabla''^{\frac{1}{2}}_0 : \Lambda(D_{\nabla''^{\frac{1}{2}}_0}) \to \mathcal{H}$:

$$\Lambda(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U,\eta_U})(U) d\mu(U)^*) \mapsto \Lambda(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U^{-1}\xi_U, E_U\eta_U})(U) d\mu(U)^*).$$

Then $\nabla''^{\frac{1}{2}}_0$ is a densely defined, preclosed operator and its closure, denoted by $\nabla''^{\frac{1}{2}}$, is a self-adjoint, strictly positive operator.

Moreover, the polar decomposition of T'' is given by $T'' = J''\nabla''^{\frac{1}{2}}$.

We now assume that (M, Δ) is unimodular. So we find $T' = T'' = T$. Theorem 2.2.8 gives an explicit expression for the modular operator and modular conjugation. The unimodulairty implies that the operator $\nabla'^{\frac{1}{2}}$ equals $\nabla^{\frac{1}{2}}$. Therefore, we know from Tomita-Takesaki theory [75] that

$$\nabla'^{it}\Gamma(x) = \Gamma(\sigma_t(x)), \qquad x \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*. \tag{2.10}$$

We use this to obtain the formula for the modular automorphism group σ.

Theorem 2.2.10. *Suppose that (M, Δ) is unimodular. Let $(\xi_U)_U, (\eta_U)_U$ be square integrable vector fields. The modular automorphism group σ_t of the Haar weight φ can be expressed as:*

$$\sigma_t \left(\int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U). \tag{2.11}$$

Proof. For $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$, $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$, such that $(\xi_U \otimes \overline{\eta_U})_U$ is a square integrable field of vectors and $\eta \in \mathrm{Dom}(E)$, we find

$$\nabla'^{it}\Gamma \left(\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \Gamma \left(\int (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U) \right). \tag{2.12}$$

Indeed, $\left(\int^\oplus (D_U \otimes \overline{E_U^{-1}}) d\mu(U) \right)^{2it} (\xi \otimes \overline{\eta}) = \int^\oplus (D_U^{2it}\xi_U \otimes \overline{E_U^{2it}\eta_U}) d\mu(U)$ by Theorem A.1.13, so (2.12) follows from Lemma 2.1.4 and Proposition 2.2.6. Now, (2.10) together with the injectivity of Γ yield

$$\sigma_t \left(\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \int (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U). \tag{2.13}$$

Now let $\eta = \int^\oplus \eta_U d\mu(U) \in \mathcal{H}$ and $\xi = \int^\oplus \xi_U d\mu(U) \in \mathcal{H}$ be arbitrary. We take sequences of square integrable vector fields $\xi_n = \int^\oplus \xi_{U,n} d\mu(U)$, $\eta_n = \int^\oplus \eta_{U,n} d\mu(U)$ such that $(\xi_{U,n} \otimes \overline{\eta_{U,n}})_U$ is a square integrable field of vectors, $\eta_n \in \mathrm{Dom}(E)$ and such that ξ_n converges to ξ and η_n converges to η. Then $\int (\iota \otimes \omega_{\xi_{U,n}, \eta_{U,n}})(U) d\mu(U)$ is σ-weakly convergent to $\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U)$ and hence

$$\sigma_t \left(\int (\iota \otimes \omega_{\xi_U, \eta_U})(U) d\mu(U) \right) = \lim_{n \to \infty} \sigma_t \left(\int (\iota \otimes \omega_{\xi_{U,n}, \eta_{U,n}})(U) d\mu(U) \right) =$$

$$\lim_{n \to \infty} \left(\int (\iota \otimes \omega_{D_U^{2it}\xi_{U,n}, E_U^{2it}\eta_{U,n}})(U) d\mu(U) \right) = \int (\iota \otimes \omega_{D_U^{2it}\xi_U, E_U^{2it}\eta_U})(U) d\mu(U),$$

which yields (2.11). □

We used (2.10) to obtain (2.11). The unimodularity assumption is essential for Theorem 2.2.10.

Remark 2.2.11. In the non-unimodular case one is able to obtain a formula similar to (2.11). In that case, the modular operator δ and scaling constant ν play a role. One gets,

$$\begin{aligned}\sigma_t^\psi\left(\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)\right) &= \nu^{\frac{1}{2}it^2}\delta^{it}\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{D_U^{2it}\xi_U,E_U^{2it}\eta_U})(U)d\mu(U),\\ \sigma_t\left(\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)\right) &= \nu^{\frac{1}{2}it^2}\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{D_U^{2it}\xi_U,E_U^{2it}\eta_U})(U)d\mu(U)\delta^{it}.\end{aligned} \tag{2.14}$$

Let us sketch how to prove the first line of (2.14). It follows from an analysis of the proof of Connes' cocycle derivative theorem, which involves a 2×2-matrix trick [75, Theorem VIII.3.3]. Let ∇_ψ be the modular operator of ψ. Fix $x\in\mathfrak{n}_\psi\cap\mathfrak{n}_\psi^*$. From general Tomita-Takesaki theory we obtain:

$$\Gamma(\sigma_t^\psi(x)) = \nabla_\psi^{it}\Gamma(x). \tag{2.15}$$

From [75, Eqn. (29) and (31)], we see that

$$\nabla_\psi^{it} = \left(\frac{D\psi}{D\varphi}\right)_t \nabla^{\prime it}. \tag{2.16}$$

Moreover, by [82, Proposition 4.3] and using the fact that $\psi\sigma_t = \nu^{-t}\psi$, we see that

$$\left(\frac{D\psi}{D\varphi}\right)_t = \nu^{\frac{1}{2}it^2}\delta^{it}. \tag{2.17}$$

Combining (2.16) and (2.17) we see that $\nabla_\psi^{it} = \nu^{\frac{1}{2}it^2}\delta^{it}\nabla^{\prime it}$. Using the fact that δ is affiliated with M as well as the explicit formula for ∇' obtained in Proposition 2.2.6, we can write (2.15) as follows. For $\eta=\int^\oplus\eta_U d\mu(U)\in\mathcal{H}$, $\xi=\int^\oplus\xi_U d\mu(U)\in\mathcal{H}$, such that $(\xi_U\otimes\overline{\eta_U})_U$ is a square integrable field of vectors and $\eta\in\mathrm{Dom}(E)$,

$$\begin{aligned}&\Gamma\left(\sigma_t^\psi\left(\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)\right)\right)\\ =&\nu^{\frac{1}{2}it^2}\delta^{it}\nabla^{\prime it}\Gamma\left(\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{\xi_U,\eta_U})(U)d\mu(U)\right)\\ =&\Gamma\left(\nu^{\frac{1}{2}it^2}\delta^{it}\int_{\mathrm{IC}(M)}(\iota\otimes\omega_{D_U^{2it}\xi_U,E_U^{2it}\eta_U})(U)d\mu(U)\right).\end{aligned}$$

The proof can be finished exactly as in Theorem 2.2.10.

Note that δ does not preserve matrix coefficients, i.e. if $U\in\mathrm{IC}(M)$ and $\xi,\eta\in\mathcal{H}_U$, then $\delta^{it}(\iota\otimes\omega_{\xi,\eta})(U)$ is not necessarily a matrix coeffcient of U. This is why we have focussed on the unimodular case in Theorem 2.2.10. In particular, it is for this reason that we can easily compute Duflo-Moore operators in Section 2.4.

2.3 $SU_q(1,1)_{\rm ext}$ satisfies the assumptions of the Plancherel theorem

In order to apply our results so far, we first we check that $SU_q(1,1)_{\rm ext}$ satisfies the assumption of the Plancherel theorem. Note that in [35] a decomposition of the left regular corepresentation is given. This suggests that the Plancherel theorem applies to (M,Δ). However, this is not automatic. The classical example of the free group with two generators illustrates that there are groups that have a (non-unique) decomposition of the left regular representation [30, Section 7.6], but the dual von Neumann algebra, given by its group von Neumann algebra is of type II [42].

Notation 2.3.1. In Sections 2.3 and 2.4, we let (M,Δ) be $SU_q(1,1)_{\rm ext}$. We introduced the relevant notation in Section 1.8. We show that $SU_q(1,1)_{\rm ext}$ satisfies the assumptions of the Plancherel theorem. Therefore, there are Duflo-Moore operators and we use the short hand notation $D_{p,x}$ for $D_{W_{p,x}}$, where $p \in q^{\mathbb{Z}}$ and $x \in [-1,1] \cup \sigma_d(\Omega_p)$. Note that μ denotes both the Plancherel measure as well as as the function on $\mathbb{C}\backslash\{0\}$ defined by $\mu(z) = (z+z^{-1})/2$. It should always be clear from the context what is meant.

Proposition 2.3.2. *Let $x \in [-1,1]$ and $x' \in \sigma_d(\Omega)$, so in particular $x \neq x'$. Then the irreducible summands of $W_{p,x}$ are all inequivalent from $W_{p,x'}$.*

Proof. This follows from the action of the Casimir operator, see (1.19), (1.28) for the principal series and for the discrete series, see [35]. The eigenvalues of Ω when restricted to $W_{p,x'}$ are contained in $\mathbb{R}\backslash[-1,1]$, whereas for $W_{p,x}$ the eigenvalues of Ω are in $[-1,1]$. □

Propositions 2.3.3 and 2.3.4 show that (M,Δ) satisfies the conditions of the Plancherel theorem, cf. Remark 1.6.4. The proof heavily depends on the decomposition of the multiplicative unitary W given in [35] and makes use of the Casimir operator Ω.

Proposition 2.3.3. *$\hat{M}$ is a type I von Neumann algebra.*

Proof. We start with some preliminary remarks. The projections in $\hat{M}'$ correspond to the invariant subspaces of W and the minimal projections in $\hat{M}'$ correspond to the irreducible subspaces of W. The partial isometries in $\hat{M}'$ correspond to intertwiners of closed subcorepresentations of W.

Throughout the proof we define $P \in \hat{M}'$ to be the projection onto the space $\bigoplus_{p\in q^{\mathbb{Z}}} \int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x}$. There are no intertwiners between closed subcorepresentations of $\bigoplus_{p\in q^{\mathbb{Z}}} \int_{[-1,1]}^{\oplus} W_{p,x} dx$ and the direct sum $\bigoplus_{p\in q^{\mathbb{Z}}} \int_{x\in\sigma_d(\Omega)}^{\oplus} W_{p,x}$, see Proposition 2.3.2. Therefore, P commutes with every partial isometry in $\hat{M}'$ so that P is central. We have $\hat{M}' = P\hat{M}'P \oplus (1-P)\hat{M}'(1-P)$. The von

Neumann algebra $(1-P)\hat{M}'(1-P)$ is of type I since the direct sum decomposition $\bigoplus_{p \in q^{\mathbb{Z}}} \int_{x \in \sigma_d(\Omega)}^{\oplus} W_{p,x}$ together with the preliminary remarks yield that every projection majorizes a minimal projection.

Now we prove that $P\hat{M}'P$ is a type I von Neumann algebra. Define the Hilbert spaces

$$\mathcal{L}_x = \left(\bigoplus_{p \in q^{\mathbb{Z}}} \mathcal{L}_{p,x} \right) \oplus \left(\bigoplus_{p \in q^{\mathbb{Z}}} \mathcal{L}_{p,-x} \right), \quad x \in (0,1); \qquad \mathcal{L}_0 = \oplus_{p \in q^{\mathbb{Z}}} \mathcal{L}_{p,0}.$$

Then,

$$P\mathcal{K} = \int_{[0,1]}^{\oplus} \mathcal{L}_x dx, \tag{2.18}$$

and we let $\mathcal{Z}$ denote the diagonalizable operators with respect to this direct integral decomposition.

We claim that $\mathcal{Z} \subseteq \hat{M}' \subseteq \mathcal{Z}'$. For the former inclusion, note that the step-functions in $\mathcal{Z}$ are linear combinations of projections onto invariant subspaces for $\hat{M}$. By the preliminary remarks we find $\mathcal{Z} \subseteq \hat{M}'$. To prove that $\hat{M}' \subseteq \mathcal{Z}'$, note that by [35, Corollary 4.11], $\hat{M}'$ is the σ-strong-$*$ closure of the linear span of elements $\hat{J}Q(p_1,p_2,n)\hat{J}$, $p_1, p_2 \in q^{\mathbb{Z}}$, $n \in \mathbb{Z}$. The operators $Q(p_1,p_2,n)$ are decomposable with respect to the direct integral decomposition (2.18) as proved in [35]; combine [35, Proposition 10.5] together with the direct integral decomposition [35, Theorem 5.7] and the definition of $Q(p_1,p_2,n)$ [35, Equation (20)]. We prove that $\hat{J}$ is a decomposable operator with respect to (2.18). It suffices to show that $\hat{J} \subseteq \mathcal{Z}'$ [21, Theorem II.2.1].

Let $B \subseteq [0,1]$ be a Borel set and let $P_B \in \mathcal{Z}$ be the operator defined as $P_B = \int_{[0,1]}^{\oplus} \chi_B(x) 1_{\mathcal{L}_x} dx$, where χ_B is the indicator function on B. P_B is a projection and we have

$$\begin{aligned}
&\chi_{B \cup -B}(\Omega)\mathcal{K} = \chi_{B\cup -B}(\Omega) \bigoplus_{p,m,\epsilon,\eta} \mathcal{K}(p,m,\epsilon,\eta) \\
=& \bigoplus_{p} \left(\bigoplus_{m,\epsilon\eta=1} \int_{x \in B \cup -B}^{\oplus} \mathbb{C} dx \oplus \bigoplus_{m,\epsilon\eta=-1} \int_{-x \in B\cup -B}^{\oplus} \mathbb{C} dx \right) \\
=& \bigoplus_{p} \bigoplus_{m,\epsilon,\eta} \int_{x \in B \cup -B}^{\oplus} \mathbb{C} dx = \bigoplus_{p} \int_{x \in B\cup -B}^{\oplus} \mathcal{L}_{p,x} dx = \int_{x \in B}^{\oplus} \mathcal{L}_x dx = P_B \mathcal{K},
\end{aligned} \tag{2.19}$$

where the second equation uses Theorem A.1.13 and the fact that there is a direct integral decomposition $\mathcal{K}(p,m,\epsilon,\eta) = \int_{\sigma(\Omega)}^{\oplus} \mathbb{C} dx$ for which we have $\chi_B(\Omega)\mathcal{K}(p,m,\epsilon,\eta) = \int_{\epsilon\eta x \in B}^{\oplus} \mathbb{C} dx$, see [35, Theorem 8.13]. Other equations are a matter of changing the order and combining direct integrals.

Note that Ω leaves the spaces $\mathcal{K}^+$ and $\mathcal{K}^-$ invariant. Let P^+ and P^- be the projections onto respectively $\mathcal{K}^+$ and $\mathcal{K}^-$. Write, again using the notation of

Definition 1.8.5,

$$\Omega = \begin{pmatrix} \Omega^+ & 0 \\ 0 & \Omega^- \end{pmatrix}, \qquad \Omega_0 = \begin{pmatrix} \Omega_0^+ & 0 \\ 0 & \Omega_0^- \end{pmatrix},$$

where $\Omega^\pm = \Omega P^\pm$ and $\Omega_0^\pm = \Omega_0 P^\pm$. Note that $\Omega^\pm$ is a self-adjoint extension of $\Omega_0^\pm$. By (1.30) we see that $\hat{J}$ leaves the spaces $\mathcal{K}^+$ and $\mathcal{K}^-$ invariant. We claim that

$$\hat{J}|_{\mathcal{K}^+}\Omega^+\hat{J}|_{\mathcal{K}^+} = \Omega^+, \qquad \hat{J}|_{\mathcal{K}^-}\Omega^-\hat{J}|_{\mathcal{K}^-} = -\Omega^-. \tag{2.20}$$

By Definitions 1.8.5 and (1.30) we find that $\hat{J}\Omega_0\hat{J}f_{m,p,t} = \mathrm{sgn}(pt)\Omega_0 f_{m,p,t}$, so that $\hat{J}\Omega_0^+\hat{J} = \Omega_0^+$ and $\hat{J}\Omega_0^-\hat{J} = -\Omega_0^-$. Hence $\hat{J}\Omega^+\hat{J} \supseteq \Omega_0^+$, and $\hat{J}\Omega^-\hat{J} \supseteq -\Omega_0^-$. Let $x \in \hat{M}'$, and write:

$$\hat{J}x\hat{J} = y^+ \oplus y^-, \qquad y^+ = \begin{pmatrix} y_1^+ & 0 \\ 0 & y_2^+ \end{pmatrix} \in M_+,\ y^- = \begin{pmatrix} 0 & y_2^- \\ y_1^- & 0 \end{pmatrix} \in M_-,$$

where the decomposition is as in [35, Proposition 4.8]. By that same proposition, we find that $y_1^-\Omega^+ \subseteq -\Omega^-y_1^-$, $y_2^-\Omega^- \subseteq -\Omega^+y_2^-$, $y_1^+\Omega^+ \subseteq \Omega^+y_1^+$ and $y_2^+\Omega^- \subseteq \Omega^-y_2^+$. This implies the inclusion in the following computation:

$$\begin{aligned}
&x\hat{J}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}\hat{J} = \hat{J}\hat{J}x\hat{J}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}\hat{J} \\
=&\hat{J}\begin{pmatrix} y_1^+ & 0 \\ 0 & y_2^+ \end{pmatrix}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}\hat{J} \oplus \hat{J}\begin{pmatrix} 0 & y_2^- \\ y_1^- & 0 \end{pmatrix}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}\hat{J} \\
\subseteq&\hat{J}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}(y^+ \oplus y^-)\hat{J} = \hat{J}\begin{pmatrix} \Omega^+ & 0 \\ 0 & -\Omega^- \end{pmatrix}\hat{J}x.
\end{aligned}$$

So $\hat{J}\Omega^+\hat{J} \oplus -\hat{J}\Omega^-\hat{J}$ is a self-adjoint operator affiliated to $\hat{M}$ extending Ω_0. So [35, Theorem 4.6] implies that $\left(\hat{J}|_{\mathcal{K}^+}\Omega^+\hat{J}|_{\mathcal{K}^+} \oplus -\hat{J}|_{\mathcal{K}^-}\Omega^-\hat{J}|_{\mathcal{K}^-}\right) = \Omega$, which results in (2.20).

To prove that $\hat{J} \subseteq \mathcal{Z}'$, it suffices to prove that for all Borel sets $B \subseteq [0,1]$, $\hat{J}P_B\hat{J} = P_B$. Indeed we have

$$\begin{aligned}
&\hat{J}P_B\hat{J} = \hat{J}\chi_{B\cup -B}(\Omega)\hat{J} = \hat{J}|_{\mathcal{K}^+}\chi_{B\cup -B}(\Omega^+)\hat{J}|_{\mathcal{K}^+} \oplus \hat{J}|_{\mathcal{K}^-}\chi_{B\cup -B}(\Omega^-)\hat{J}|_{\mathcal{K}^-} \\
=&\chi_{B\cup -B}(\Omega^+) \oplus \chi_{B\cup -B}(\Omega^-) = \chi_{B\cup -B}(\Omega) = P_B.
\end{aligned}$$

The first and last equality are due to (2.19); the third equality is due to (2.20). In all, we have proved that $\mathcal{Z} \subseteq \hat{M} \subseteq \mathcal{Z}'$.

Let $W_x = \left(\bigoplus_{p\in q^{\mathbb{Z}}} W_{p,x}\right) \oplus \left(\bigoplus_{p\in q^{\mathbb{Z}}} W_{p,-x}\right)$ for $x \in (0,1]$ and set furthermore $W_0 = \bigoplus_{p\in q^{\mathbb{Z}}} W_{p,0}$. The operators $Q(p_1,p_2,n)$ form a countable family that generates $\hat{M}$ [35, Proposition 4.9]. We apply [21, Theorem II.3.2] and its subsequent remark, together with [21, Theorem II.3.1] to conclude that

$$P\hat{M}P = \int_{x\in[0,1]}^{\oplus} \hat{M}_x dx, \tag{2.21}$$

where $\hat{M}_x$ is generated by $\{(\omega \otimes \iota)(W_x) \mid \omega \in M_*\}$ almost everywhere. The projections in $\hat{M}'_x$ correspond to irreducible subspaces of W_x. Since W_x decomposes as a direct sum of irreducible corepresentations [35, Proposition 5.4], every projection in $\hat{M}'_x$ majorizes a minimal projection. We find that $\hat{M}'_x$ is type I and by [42, Theorem 14.1.21], [74, Corollary V.2.24] and (2.21) we conclude that $P\hat{M}P$ is type I . □

Proposition 2.3.4. *$\hat{M}_c$ is separable.*

Proof. Note that if $\omega_n \in M_*$ is sequence that converges in norm to $\omega \in M_*$, then $\|\lambda(\omega_n) - \lambda(\omega)\| \leq \|\omega_n - \omega\|$ so that $\lambda(\omega_n)$ converges in norm to $\lambda(\omega)$. Since the norm on $\hat{M}_c$ is the operator norm on the GNS-space and $\hat{M}_c$ is the C*-algebra obtained as the closure of $\{\lambda(\omega) \mid \omega \in M_*\}$. It suffices to check that M_* is separable. The $\mathbb{Q}$-linear span of $\{\omega_{f_{m_0,p_0,t_0},f_{m_1,p_1,t_1}} \mid m_i \in \mathbb{Z}, p_i, t_i \in I_q, i = 0, 1\}$ is weakly dense, hence norm dense in M_*. □

In particular, a Plancherel measure of (M, Δ) is given by the direct integral decomposition (1.22). It is tempting to fix μ as this choice for the Plancherel measure. However, there is a more natural choice as we see in Section 2.4.

2.4 The dual Haar weights of $SU_q(1,1)_{\rm ext}$

Using the theory of square integrable corepresentions Desmedt [19] determined the Duflo-Moore operators D_U and E_U for the corepresentions U that appear as discrete mass points of the Plancherel measure, see also Remark 1.6.5. In particular, his theory applies to compact quantum groups, for which every corepresentation is square integrable. As a non-compact example, Desmedt was able to determine the operators D_U for the discrete series corepresentations of $SU_q(1,1)_{\rm ext}$. The calculations involve summation formulae for basic hypergeometric series. Having the theory of Sections 2.1 and 2.2 at hand we determine the operators D_U and E_U for the principal series corepresentations of $SU_q(1,1)_{\rm ext}$ by a completely different method.

Notation 2.4.1. Recall the notational conventions from Notation 2.3.1. For now, we let μ be some Plancherel measure for (M, Δ). We give a specific choice at the end of this section. Every field, integral, et cetera should be understood with respect to this measure, which is absolutely continuous with respect to the measure used in (1.22).

Fix $p \in q^{\mathbb{Z}}$. The idea of finding expressions for the Duflo-Moore operators is to determine the tensor product $D_{p,x}^{it} \otimes \overline{E_{p,x}^{it}}$ for almost all $x \in [-1, 1]$. This way, $D_{p,x}$ and $E_{p,x}$ are determined up to a constant for almost all $x \in [-1, 1]$.

Theorem 2.4.2. *Consider the quantum group $SU_q(1,1)_{\mathrm{ext}}$. Fix $p \in q^{\mathbb{Z}}$ and let μ be a Plancherel measure. There is a positive μ-measurable function $d_p(x)$ with $x \in [-1,1] \cup \sigma_d(\Omega_p)$, such that*

$$\begin{aligned} D_{p,x} f_m^{\varepsilon,\eta}(p,x) &= q^m d_p(x) f_m^{\varepsilon,\eta}(p,x), \\ E_{p,x} f_m^{\varepsilon,\eta}(p,x) &= q^{-m} d_p(x) f_m^{\varepsilon,\eta}(p,x). \end{aligned}$$

The function $d_p(x)$ depends on the choice of the Plancherel measure μ, see Theorem 1.6.1.

Proof. Fix $p \in q^{\mathbb{Z}}$. Let $\varepsilon, \eta, m, \varepsilon', \eta', m'$ be μ-measurable functions of $x \in [-1,1]$, thus $\varepsilon = \varepsilon(x), \eta = \eta(x), \ldots$. Moreover, let $\varepsilon, \eta, \varepsilon', \eta'$ take values in $\{-,+\}$ and let m, m' take values in $\mathbb{Z} + \frac{1}{2}\chi(p)$. Let f, f' be μ-square integrable complex functions on $[-1,1]$. Then $f(x) f_m^{\varepsilon,\eta} = f(x) f_m^{\varepsilon,\eta}(p,x)$ and $f'(x) f_{m'}^{\varepsilon',\eta'} = f'(x) f_{m'}^{\varepsilon',\eta'}(p,x)$ are μ-square integrable fields of vectors. Since the modular automorphism group σ_t is implemented by $\gamma^*\gamma$ (by construction of φ, or see [51, Section 4]), Theorem 2.2.10 yields

$$\begin{aligned} &\left(\int_{[-1,1]} (\iota \otimes \omega_{f(x) D_{p,x}^{2it} f_m^{\varepsilon,\eta}, f'(x) E_{p,x}^{2it} f_{m'}^{\varepsilon',\eta'}}) (W_{p,x}) \, d\mu(x) \right) f_{m_0,p_0,t_0} \\ =&\sigma_t \left(\int_{[-1,1]} (\iota \otimes \omega_{f(x) f_m^{\varepsilon,\eta}, f'(x) f_{m'}^{\varepsilon',\eta'}}) (W_{p,x}) \, d\mu(x) \right) f_{m_0,p_0,t_0} \\ =&|\gamma|^{2it} \left(\int_{[-1,1]} (\iota \otimes \omega_{f(x) f_m^{\varepsilon,\eta}, f'(x) f_{m'}^{\varepsilon',\eta'}}) (W_{p,x}) \, d\mu(x) \right) |\gamma|^{-2it} f_{m_0,p_0,t_0} \\ =& \left(\frac{p_0^2}{p_0^2 q^{-2m-2m'}} \right)^{it} \int_{[-1,1]} f(x) \overline{f'(x)} \\ &\times C(\eta\varepsilon x; m' - 1/2\chi(p), \varepsilon', \eta'; \varepsilon\varepsilon' |p_0| q^{-m-m'}, p_0, m - m') \\ &\times \delta_{sgn(p_0),\eta\eta'} f_{m_0 - m + m', \varepsilon\varepsilon' |p_0| q^{-m-m'}, t_0} \, d\mu(x) \\ =&q^{it(2m+2m')} \int_{[-1,1]} (\iota \otimes \omega_{f(x) f_m^{\varepsilon,\eta}, f'(x) f_{m'}^{\varepsilon',\eta'}}) (W_{p,x}) \, d\mu(x) f_{m_0,p_0,t_0}. \end{aligned} \tag{2.22}$$

Define A and B as the unbounded self-adjoint operators on $\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)$ determined by

$$\begin{aligned} A &= \int_{[-1,1]}^{\oplus} A_{p,x} d\mu(x), \qquad A_{p,x} f_m^{\varepsilon,\eta}(p,x) = p q^{2m} f_m^{\varepsilon,\eta}(p,x). \\ B &= \int_{[-1,1]}^{\oplus} B_{p,x} d\mu(x), \qquad B_{p,x} f_m^{\varepsilon,\eta}(p,x) = p q^{-2m} f_m^{\varepsilon,\eta}(p,x). \end{aligned}$$

In the remainder of the proof, we show that (2.22) implies that $D_{p,x} \otimes \overline{E_{p,x}} = A_{p,x}^{\frac{1}{2}} \otimes \overline{B_{p,x}}^{\frac{1}{2}}$ for almost every $x \in [-1,1]$. The proof is subtle and relies on the results of Section 2.1.

Note that (2.22) yields

$$\begin{aligned}&\int \left(\iota \otimes \omega_{f(x)D_{p,x}^{2it}f_m^{\varepsilon,\eta},f'(x)E_{p,x}^{2it}f_{m'}^{\varepsilon',\eta'}}\right)(W_{p,x})\,d\mu(x)\\ &=\int \left(\iota \otimes \omega_{f(x)A_{p,x}^{it}f_m^{\varepsilon,\eta},f'(x)B_{p,x}^{it}f_{m'}^{\varepsilon',\eta'}}\right)(W_{p,x})\,d\mu(x),\end{aligned} \tag{2.23}$$

where the integrals are taken over $[-1,1]$. For any two bounded operators $F = \int_{[-1,1]}^{\oplus} F_{p,x} d\mu(x)$, $G = \int_{[-1,1]}^{\oplus} G_{p,x} d\mu(x)$ on $\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)$, the linear mapping

$$\left(\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)\right) \otimes \overline{\left(\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)\right)} \to M$$

given by

$$v \otimes \overline{w} = \int_{[-1,1]}^{\oplus} v_x d\mu(x) \otimes \overline{\int_{[-1,1]}^{\oplus} w_x d\mu(x)} \mapsto \int_{[-1,1]} \left(\iota \otimes \omega_{F_{p,x}v_x, G_{p,x}w_x}\right)(W_{p,x})\,d\mu(x)$$

is norm-σ-weakly continuous since

$$\left|\int_{[-1,1]} \alpha \otimes \omega_{v_x,w_x}(W_{p,x}) d\mu(x)\right| \leq \|\alpha\|\|F\|\|G\|\|v\|\|w\|, \qquad \alpha \in M_*.$$

Therefore, for $v = \int_{[-1,1]}^{\oplus} v_x d\mu(x), w = \int_{[-1,1]}^{\oplus} w_x d\mu(x) \in \int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)$, using [21, II.1.6, Proposition 7] and (2.23),

$$\int_{[-1,1]} \left(\iota \otimes \omega_{D_{p,x}^{2it}v_x, E_{p,x}^{2it}w_x}\right)(W_{p,x})\,d\mu(x) = \int_{[-1,1]} \left(\iota \otimes \omega_{A_{p,x}^{it}v_x, B_{p,x}^{it}w_x}\right)(W_{p,x})\,d\mu(x). \tag{2.24}$$

For $v = \int_{[-1,1]}^{\oplus} v_x d\mu(x), w = \int_{[-1,1]}^{\oplus} w_x d\mu(x) \in \int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)$, with $(v_x)_x$ essentially bounded, $w \in \mathrm{Dom}\left(\int_{[-1,1]}^{\oplus} E_{p,x} d\mu(x)\right)$, Theorem 2.1.5 implies that

$$\int_{[-1,1]} \left(\iota \otimes \omega_{D_{p,x}^{2it}v_x, E_{p,x}^{2it}w_x}\right)(W_{p,x})\,d\mu(x) \in \mathfrak{n}_{\psi}.$$

By (2.24) and Theorem 2.1.6, $B_{p,x}^{it}w_x \in \mathrm{Dom}(E_{p,x})$ almost everywhere in the support of $(v_x)_x$. Theorem 2.1.5 implies that for $v' = \int_{[-1,1]}^{\oplus} v'_x d\mu(x), w' = \int_{[-1,1]}^{\oplus} w'_x d\mu(x) \in \int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)$ with the additional assumptions given by

$w' \in \mathrm{Dom}\left(\int_{[-1,1]}^{\oplus} E_{p,x}^2 d\mu(x)\right)$ and $(v'_x \otimes E_{p,x}w'_x)_x$ is square integrable,

$$\begin{aligned}
&\int_{[-1,1]} \langle B_{p,x}^{it} w_x, E_{p,x}^2 w'_x\rangle \langle v'_x, A_{p,x}^{it} v_x\rangle d\mu(x)\\
=&\psi\left(\left(\int_{[-1,1]} (\iota \otimes \omega_{A_{p,x}^{it} v_x, B_{p,x}^{it} w_x})(W_{p,x}) d\mu(x)\right)^* \int_{[-1,1]} (\iota \otimes \omega_{v'_x, w'_x})(W_{p,x}) d\mu(x)\right)\\
=&\psi\left(\left(\int_{[-1,1]} (\iota \otimes \omega_{D_{p,x}^{2it} v_x, E_{p,x}^{2it} w_x})(W_{p,x}) d\mu(x)\right)^* \int_{[-1,1]} (\iota \otimes \omega_{v'_x, w'_x})(W_{p,x}) d\mu(x)\right)\\
=&\int_{[-1,1]} \langle E_{p,x}^{2it} w_x, E_{p,x}^2 w'_x\rangle \langle v'_x, D_{p,x}^{2it} v_x\rangle d\mu(x).
\end{aligned}$$

We know that $E_{p,x}$ is strictly positive by the Plancherel theorem. The elements $\int_{[-1,1]}^{\oplus} v'_x \otimes \overline{E_{p,x}^2 w'_x} d\mu(x)$ are dense in $\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x) \otimes \overline{\int_{[-1,1]}^{\oplus} \mathcal{L}_{p,x} d\mu(x)}$, so $\int_{[-1,1]}^{\oplus} D_{p,x}^{2it} \otimes \overline{E_{p,x}^{2it}} d\mu(x) = \int_{[-1,1]}^{\oplus} A_{p,x}^{it} \otimes \overline{B_{p,x}^{it}} d\mu(x)$. By Stone's theorem and Theorem A.1.13 $\int_{[-1,1]}^{\oplus} D_{p,x} \otimes \overline{E_{p,x}} d\mu(x) = \int_{[-1,1]}^{\oplus} A_{p,x}^{\frac{1}{2}} \otimes \overline{B_{p,x}}^{\frac{1}{2}} d\mu(x)$. Hence, $D_{p,x} \otimes \overline{E_{p,x}}$ equals $A_{p,x}^{\frac{1}{2}} \otimes \overline{B_{p,x}}^{\frac{1}{2}}$ for almost all $x \in [-1,1]$. The theorem now follows for the principal series corepresentations. For the discrete series, the result was already obtained in [19]. Any other corepresentation is not contained in the support of the Plancherel measure [19, Theorem 3.4.8]. This concludes the theorem. $\square$

Remark 2.4.3. Desmedt [19, Section 3.5] obtains a similar result using summation formulas for basic hypergeometric series, a method different from the one presented here. Note that the present method also applies to discrete series corepresentations and avoids caclulations involving special functions.

In the remainder of this section, we will determine the function $d_p(x)$ for $p \in q^{\mathbb{Z}}$ and $x \in \sigma_d(\Omega_p)$ for a fixed Plancherel measure. The essential ingredient will be [35, Theorem C.1], which considers an integral transformation with the kernel being a special case of the little q-Jacobi functions. Therefore, we fix a Plancherel measure here by defining μ as a special case of the Askey-Wilson measure restricted to the interval $[-1,1]$.

Remark 2.4.4. Since we only consider the principal series corepresentations, we restrict the Askey-Wilson measure to the interval $[-1,1]$. Note that this is precisely the continuous part of the support of the Askey-Wilson measure [52]. A priori one would think that it is possible to simultaneously take the discrete and principal series into account in our calculation. However, the support of the Askey-Wilson measure and the discrete mass points of the Plancherel measure, i.e. $\sigma_d(\Omega_p)$ do not match. So in that case a matching of these two sets is necessary. We do not know if this is indeed possible. However, since the Duflo-Moore operators were already computed for the discrete series we restrict ourselves to the continous part of the Plancherel measrue.

Consider the Askey-Wilson measure restricted to the interval $[-1,1]$ [1, Section 2], [32, Chapter 6]:

$$\int_{-1}^{1} f(x) d\nu(x; a, b, c, d \mid q) = \frac{1}{h_0 2\pi} \int_0^{\pi} f(\cos\theta) w(e^{i\theta}) d\theta.$$

Here we used the notation $w(z) = w(z; a, b, c, d \mid q), h_0 = h_0(a, b, c, d \mid q)$ and

$$\begin{aligned} h_0(a, b, c, d \mid q) &= \frac{(abcd; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}, \\ w(z; a, b, c, d \mid q) &= \frac{(z^2, z^{-2}; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}. \end{aligned}$$

Now consider the measure ν:

$$\int_{-1}^{1} f(x) d\nu(x; a, b; d \mid q) = h_0(a, b, q/d, d \mid q) \int_{-1}^{1} f(x) dm(x; a, b, q/d, d \mid q).$$

Finally set:

$$d\mu(\cdot) = d\nu(\cdot; q/p, pq, -p/q \mid q^2). \tag{2.25}$$

Note that μ is equivalent to the Lebesgue measure on the interval $[-1,1]$ and therefore is a suitable choice for the Plancherel measure restricted to the principal series corepresentations.

The following result is crucial for computing the corresponding Duflo-Moore operators.

Corollary 2.4.5 (Theorem C.1 of [35]). *Let $d, z \in \mathbb{C}$ such that $dz \in \mathbb{R}$ and $z, d^2 z \notin q^{\mathbb{Z}}$. Let $c = q^2$. Assume that one of the following three conditions holds: (1) $\bar{z}c = d^2 z$, or (2) $z > 0, c \neq d^2$ and $zq^{k_0+1} < c/d^2 < zq^{k_0}$, where $k_0 \in \mathbb{Z}$ is such that $1 < q^{k_0} z < q^{-1}$, or (3) $z < 0, c \neq d^2$ and $c/d^2 > 0$. Define the operator:*

$$\begin{aligned} &\mathcal{G} : L^2([-1,1], d\nu) \to l^2(\mathbb{Z}) : \\ &f \mapsto \int_{-1}^{1} f(x)(c, z, q^2/z; q^2)_\infty w(k)\, {}_2\varphi_1 \left(\begin{matrix} dy, d/y \\ c \end{matrix} ; q^2, zq^{2k} \right) d\nu(x), \end{aligned}$$

where we use the notation:

$$w(k) = d^k \sqrt{\frac{(cq^{2-2k}/d^2 z; q^2)_\infty}{(q^{2-2k}/z; q^2)_\infty}}, \qquad d\nu(x) = d\nu(x; c/d, d; q/dz \mid q).$$

Then $\mathcal{G}$ is a partial isometry with initial space $L^2([-1,1], d\nu)$.

Proof: Let $\mathcal{F}_T$ be as in [35, Theorem C.1]. By that same theorem $\mathcal{F}_T$ is unitary. Then $\mathcal{G}$ is the adjoint of $\mathcal{F}_T$ restricted to $L^2([-1,1], d\nu)$. $\square$

Theorem 2.4.6. *Consider the quantum group $SU_q(1,1)_{\mathrm{ext}}$ and fix the Plancherel measure μ as discribed above. Fix $p \in q^{\mathbb{Z}}$. For $x \in [-1,1]$, the corresponding Duflo-Moore operators are given by:*

$$\begin{aligned} D_{x,p} f_m^{\varepsilon,\eta}(p,x) &= q^m q^{\frac{1}{2}(\chi(p)^2 - 2\chi(p)+4)} (q^2;q^2)_\infty c_q^2 f_m^{\varepsilon,\eta}(p,x), \\ E_{x,p} f_m^{\varepsilon,\eta} &= q^{-m} q^{\frac{1}{2}(\chi(p)^2 - 2\chi(p)+4)} (q^2;q^2)_\infty c_q^2 f_m^{\varepsilon,\eta}(p,x), \end{aligned} \tag{2.26}$$

for almost every $x \in [-1,1]$. That is, the undetermined function in Theorem 2.4.2 is constant, say d_p, and equals

$$d_p = d_p(x) = q^{\frac{1}{2}(\chi(p)^2 - 2\chi(p)+4)} (q^2;q^2)_\infty c_q^2.$$

Proof. In the computations below every integral is taken over the interval $[-1,1]$ with respect to the above choice of the Askey-Wilson measure μ. $p \in q^{\mathbb{Z}}$ is fixed. We fix as well $\varepsilon, \eta \in \{-1,1\}$. Let f and g be μ-square integrable functions on $[-1,1]$. By Theorems 2.1.5 and 2.4.2, we get the equations:

$$\begin{aligned} &\varphi\left(\int \left(\iota \otimes \omega_{f(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x}\right) d\mu(x)^* \right. \\ &\quad \left. \times \int \left(\iota \otimes \omega_{g(x') f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x'}\right) d\mu(x') \right) \\ &= \int \langle D_{p,x} f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), D_{p,x} f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x) \rangle \\ &\quad \times \langle g(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x) \rangle d\mu(x) \\ &= \int p |d_p(x)|^2 g(x) \overline{f(x)} d\mu(x). \end{aligned} \tag{2.27}$$

For the left hand side of (2.27) we make the following computation. Here, we use the expression for the left Haar weight φ of (M,Δ), see (1.17) and the expression for the matrix coefficients of the multiplicative unitary contained in (1.24).

$$\begin{aligned} &\varphi\left(\int \left(\iota \otimes \omega_{f(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x}\right) d\mu(x)^* \right. \\ &\left. \int \left(\iota \otimes \omega_{g(x') f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x'}\right) d\mu(x') \right) \\ &= \sum_{p_0 \in I_q} p_0^{-2} \langle \int \left(\iota \otimes \omega_{g(x') f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x'}\right) d\mu(x') f_{0,p_0,p_0}, \\ &\int \left(\iota \otimes \omega_{f(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right) \left(W^*_{p,x}\right) d\mu(x) f_{0,p_0,p_0} \rangle \\ &= \sum_{p_0 \in q^{\mathbb{Z}}} p_0^{-2} \int \int g(x') \overline{f(x)} C(\varepsilon\eta x'; 0, \varepsilon, \eta; p_0, |p_0| p, 0) \\ &\times \overline{C(\varepsilon\eta x; 0, \varepsilon, \eta; p_0, |p_0| p, 0)} d\mu(x) d\mu(x'). \end{aligned} \tag{2.28}$$

Set $p_0 = q^k$ and $x = \mu(\lambda)$ with $\lambda \in \mathbb{T}$. Assume for the moment that $\varepsilon\eta = -1$. Then, using the explicit formula for the C-function (1.25) and the θ-product identity (1.16), we find:

$$\begin{aligned}
&p_0^{-1}C(\varepsilon\eta x; 0, \varepsilon, \eta; p_0, |p_0|p, 0)\\
=&p_0 p\nu(p_0)\nu(p_0 p)\sqrt{(-p_0^2, -p_0^2 p^2; q^2)_\infty}\\
&\times (-q^2/p_0^2p^2; q^2)_\infty c_q^2 (q^2;q^2)_\infty^2 \, {}_2\varphi_1\left(\begin{matrix} pq/\lambda, p\lambda q \\ q^2 \end{matrix}; q^2, -q^2/p^2p_0^2\right)\\
=&p_0 p\nu(p_0)\nu(p_0 p) p^{-2k} q^{-k(k-1)}(-p^2, -q^2/p^2; q^2)_\infty\\
&\times \sqrt{(-p_0^2;q^2)_{\chi(p)}} c_q^2 (q^2;q^2)_\infty^2 \, {}_2\varphi_1\left(\begin{matrix} pq/\lambda, p\lambda q \\ q^2 \end{matrix}; q^2, -q^2/p^2p_0^2\right)\\
=&q^{\frac{1}{2}(\chi(p)^2-\chi(p)+4)} c_q^2\\
&\times (q^2;q^2)_\infty (c, z, q^2/z; q^2)_\infty w(-k) \, {}_2\varphi_1\left(\begin{matrix} dy, d/y \\ c \end{matrix}; q^2, zq^{-2k}\right),
\end{aligned}$$

where

$$\begin{matrix} d = pq, & y = \lambda, \\ c = q^2, & z = -q^2/p^2, \end{matrix} \quad w(k) = d^k \sqrt{\frac{(cq^{2-2k}/d^2 z; q^2)_\infty}{(q^{2-2k}/z;q^2)_\infty}} = p^k q^k \sqrt{(-q^{-2k};q^2)_{\chi(p)}}.$$

Note that $z, d^2 z \notin q^{\mathbb{Z}}$ and $c = q^2$. Furthermore, either condition (1) or (3) of Corollary 2.4.5 holds depending on whether $p = 1$ or $p \neq 1$. Then, according to this corollary, continuing equation (2.28):

$$\begin{aligned}
&\varphi\left(\int \left(\iota \otimes \omega_{f(x) f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right)\left(W^*_{p,x}\right) d\mu(x)^* \right.\\
&\left.\int \left(\iota \otimes \omega_{g(x') f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x), f^{\varepsilon,\eta}_{1/2\chi(p)}(p,x)}\right)\left(W^*_{p,x'}\right) d\mu(x')\right)\\
=&(q^{\frac{1}{2}(\chi(p)^2-\chi(p)+4)}(q^2;q^2)_\infty c_q^2)^2 \langle \mathcal{G}(g), \mathcal{G}(f)\rangle_{l^2(\mathbb{Z})}\\
=&(q^{\frac{1}{2}(\chi(p)^2-\chi(p)+4)}(q^2;q^2)_\infty c_q^2)^2 \int_{-1}^{1} g(x)\overline{f}(x) d\mu(x)
\end{aligned} \tag{2.29}$$

Equating this to the right hand side of (2.27) we find that:

$$d_p(x) = q^{\frac{1}{2}(\chi(p)^2-2\chi(p)+4)}(q^2;q^2)_\infty c_q^2 \tag{2.30}$$

Now, the theorem follows from Theorem 2.4.2. □

Remark 2.4.7. It is also possible to derive Theorem 2.4.6 without relying on the result of Theorem 2.4.2 using a direct computation. The computations get longer and more involved.

As a corollary, we obtain the dual Haar weights of $SU_q(1,1)_{\text{ext}}$.

Corollary 2.4.8. *The von Neumann algebra $\hat{M}$ is a subalgebra of*

$$\bigoplus_{p\in q^{\mathbb{Z}}}\left(\int_{[-1,1]}^{\oplus} B(\mathcal{L}_{p,x})d\mu(x)\oplus\bigoplus_{x\in\sigma_d(\Omega_p)} B(\mathcal{L}_{p,x})\right).$$

Let $A = A(p,x), p\in q^{\mathbb{Z}}, x\in\sigma_d(\Omega_p)$ be an element of M^+. Consider the weights given by:

$$\begin{aligned}\phi_{\pm}(A) = \sum_{p\in q^{\mathbb{Z}}}\Bigg(\int_{[-1,1]} &\sum_{\substack{m\in\mathbb{Z}+1/2\chi(p),\\ \varepsilon,\eta\in\{-1,1\}}} d_p^{-2} q^{\mp 2m}\langle A(p,x)f_m^{\varepsilon,\eta}(p,x), f_m^{\varepsilon,\eta}(p,x)\rangle d\mu(x)\\ &+\sum_{x\in\sigma_d(\Omega_p)}\sum_{m\in\mathbb{Z}}\left(\frac{2q^{\pm m+\chi(p)+l(p,x))}}{\sqrt{q^{2(1+\chi(p)+2l(p,x)}-1}}\right)^{-2}\langle A(p,x)f_m^{\varepsilon,\eta}(p,x), f_m^{\varepsilon,\eta}(p,x)\rangle\Bigg),\end{aligned}$$

where $l(p,x)$ is the unique number in $\mathbb{Z}$ determined by $|x| = \mu(q^{1+2l(p,x)}p)$ and $1+2l(p,x)+\chi(p) < 0$. Then, the Haar weights on $(\hat{M},\hat{\Delta})$ are given by:

$$\hat{\varphi} = \phi_+, \qquad \hat{\psi} = \phi_-.$$

Proof. The proof directly follows from the nature of the Duflo-Moore operators. Consider the trace on $\hat{M}$ which is given by

$$\bigoplus_{p\in q^{\mathbb{Z}}}\left(\int_{[-1,1]}^{\oplus} \mathrm{Tr}_{p,x}(\,\cdot\,)d\mu(x)\oplus\bigoplus_{x\in\sigma_d(\Omega_p)} \mathrm{Tr}_{p,x}(\,\cdot\,)\right),$$

where $\mathrm{Tr}_{p,x}$ is the unique trace on $B(\mathcal{L}_{p,x})$ restricted to $\hat{M}$. Then, by Theorem A.4.2, $\hat{\varphi}$ has a Radon-Nikodym derivative D^{-2} with respect to this trace. By Theorem A.4.1 (1) it follows that D^{-2} is affiliated with $\hat{M}$ and therefore has a direct integral decomposition. Looking back at the proof of the Plancherel Theorem 1.6.1, the Duflo-Moore operators are exactly the operators in this direct integral decompositon.

For the discrete series corepresentations, the Duflo-Moore operators can be found in [19, Proposition 3.5.5]. Here, $D_{p,x}$ is denoted by $C_{p,x}$. The explicit formula of $E_{p,x}$ then follows from Theorem 2.4.2. □

Chapter 3

Locally compact Gelfand pairs

In the classical setting of locally compact groups, a *Gelfand pair* consists of a locally compact group G, together with a compact subgroup K such that the convolution algebra of bi-K-invariant L^1-functions on G is commutative. See [20] or [26] for a comprehensive introduction. Gelfand pairs give rise to spherical functions and a spherical Fourier transform which decomposes bi-K-invariant functions on G as an integral of spherical functions, see [20, Theorem 6.4.5] or [26, Théorème IV.2]. More particular, the following theorem, known as the Plancherel-Godement theorem is a corner stone in the theory. To recall the theorem we need some terminology.

Definition 3.0.1. Let (G, K) be a Gelfand pair. A bi-K-invariant function is called *positive definite* if it is of the form

$$x \mapsto \langle \pi(x)\xi, \xi \rangle,$$

for some *unitary* representation π of G and a K-invariant vector ξ. A positive definite function is called *spherical* if the corresponding representation (which is unique up to equivalence) is irreducible.

These are not at all the original definitions. However, by [26, Proposition II.1 and Section III] they are equivalent to the definitions given in [26]. Let $\mathrm{IR}(G, K)$ denote the positive definite, spherical functions on G.

Theorem 3.0.2 (Plancherel-Godement Theorem IV.2 of [26])**.** *Consider a Gelfand pair (G, K). Denote $L^p(K\backslash G/K)$ for the bi-K-invariant L^p-functions. For $f \in L^1(K\backslash G/K)$, let $\hat{f}$ be the function on $\mathrm{IR}(G, K)$ defined by:*

$$\hat{f}(\omega) = \int f(x)\omega(x)dx.$$

There exists a unique measure μ called the spherical Plancherel measure on $\mathrm{IR}(G, K)$ such that:

1. *Let $f \in L^1(K\backslash G/K)$ be a continuous, positive definite function. Then, $\hat{f}$ is integrable with respect to the spherical Plancherel measure and*

$$f(x) = \int_{\mathrm{IR}(G,K)} \omega(x)\hat{f}(\omega)d\mu(\omega).$$

2. *Let $f \in L^1(K\backslash G/K) \cap L^2(K\backslash G/K)$. Then $\hat{f}$ is square integrable with respect to the spherical Plancherel measure and*

$$\int |f(x)|^2 dx = \int_{\mathrm{IR}(G,K)} |\hat{f}(\omega)|^2 d\mu(\omega).$$

3. *The transformation $L^1(K\backslash G/K) \cap L^2(K\backslash G/K) \to L^2(\Omega, \mu) : f \mapsto \hat{f}$ extends to a unitary map.*

The *spherical Plancherel measure* is in many references just called *Plancherel measure*. The terminology is justified by the fact that if both the Plancherel measure and the spherical Plancherel measure are defined, then one can consider $\mathrm{IR}(G, K)$ as a measurable subspace of $\hat{G}$ and the spherical Plancherel measure is the restriction of the Plancherel measure.

For many examples, this decomposition is made precise [20]. The examples include the group of motions of the plane together with its diagonal subgroup and the pair $(SO_0(1, n), SO(n))$, where $SO_0(1, n)$ is the connected component of the identity of $SO(1, n)$. In particular the spherical functions are determined and one can derive product formulae for this type of functions.

Since the introduction of quantum groups, Gelfand pairs were studied in a quantum context, see for example [29], [69], [88], [89] and also the references given there. These papers consider pairs of quantum groups that are both compact. For such pairs it suffices to stay with a purely (Hopf-)algebraic approach. Under the assumption that every irreducible unitary corepresentation admits only one matrix element that is invariant under both the left and right action of the subgroup, these quantum groups are called (quantum) Gelfand pairs. Classically, this is equivalent to the commutativity assumption on the convolution algebra of bi-K-invariant elements. If the matrix coefficients form a commutative algebra one speaks of a strict (quantum) Gelfand pair. In the group setting every Gelfand pair is automatically strict and as such strictness is a purely non-commutative phenomenon.

For quantum groups, many deformations of classical Gelfand pairs do indeed form a quantum Gelfand pair that moreover is strict. As a compact example, $(SU_q(n), U_q(n-1))$ forms a strict Gelfand pair [89]. In a seperate paper [87] Vainerman introduces the quantum group of motions of the plane, together with the circle as a subgroup as an example of a Gelfand pair of which the larger quantum group is non-compact. As a result a product formula for the Hahn-Exton q-Bessel functions, also known as ${}_1\varphi_1$ q-Bessel functions or Jackson's third

q-Bessel functions, is obtained [87, p. 324, Corollary], see also [50, Corollary 6.4]. However, a comprehensive general framework of quantum Gelfand pairs in the non-compact operator algebraic setting was unavailable, because at that time a suitable definition of a non-compact quantum group had not been given.

Since we now have a comprehensive conceptual framework, one might wonder how Gelfand pairs fit in this framework. In particular, the quantum Plancherel theorem suggests to look for the general existence of a spherical Fourier transform.

From this perspective, it is a natural question if the study of Gelfand pairs can be continued in the locally compact operator algebraic setting. In this chapter we give this interpretation. Motivated by Desmedt's proof of the quantum Plancherel theorem, we define the necessary structures to obtain an analogue of the classical Plancherel-Godement theorem [20, Theorem 6.4.5] or [26, Théorème IV.2]. For this the operator algebraic interpretation of Gelfand pairs is *essential*.

We keep the setting a bit more general than one would expect. For a classical Gelfand pair of groups, one can prove that the larger group is unimodular from the commutativity assumption on bi-K-invariant elements. Here we study pairs of quantum groups for which the smaller quantum group is compact and we *assume* that the larger group is unimodular. We do not impose the classically stronger commutativity assumption. The reason for this is that we do want to study $SU_q(1,1)_{\mathrm{ext}}$ together with its diagonal subgroup. However, the natural analogue of the commutativity assumption would exclude this example.

We mention that it is known that the notion of a quantum subgroup is in a sense too restrictive. Using Koornwinder's twisted primitive elements, it is possible to define double coset spaces associated with $SU_q(2)$ and get so called (σ, τ)-spherical elements, see [49] for this particular example. See also [52] for a similar study of $SU_q(1,1)$ on an algebraic level. The subgroup setting then corresponds to the limiting case $\sigma, \tau \to \infty$. In the present paper we do not incorporate such a general setting and in fact, our techniques seem to fail in this setting.

Motivated by the Hopf-algebraic framework, we introduce the non-compact analogues of bi-K-invariant functions and its dual [29], [89] and equip these with weights. We do this in a von Neumann algebraic manner and for the dual structure also in a C*-algebraic manner. We prove that the C*-algebraic weight lifts to the von Neumann algebraic weight. Moreover, we establish a spherical analogue of Kustermans' Theorem 1.5.9, which establishes a correspondence between representations of the (universal) C*-algebraic dual quantum group and corepresentations of the quantum group itself. Eventually, this structure culminates in a quantum Plancherel-Godement theorem, as an application of Desmedt's auxiliary result, i.e. Theorem 1.6.2. This illustrates the advantage of an operator algebraic interpretation above the Hopf algebraic approach. In particular, we get a spherical L^2-Fourier transform, or spherical Plancherel transformation, and we show in principle that this is a restriction of the non-spherical Plancherel transformation.

In Chapter 4 we treat the first example of a quantum Gelfand pair involving a q-deformation of $SU(n,1)$. Namely, we treat $SU_q(1,1)_{\text{ext}}$.

The contents of this chapter are contained in the forthcoming paper [8].

This chapter has the following structure. We start with some preliminary results on homogeneous spaces and their conditional expectation values, see Section 3.1. Then, in Sections 3.2 and Section 3.3 we study the homogeneous space of left and right invariant elements. We also give their dual spaces in a von Neumann algebraic, and a reduced and universal C*-algebraic way.

Next we define weights on these spaces in Section 3.4. We prove that on the dual side we can define a C*-algebraic weight which has a W^*-lift to a natural von Neumann algebraic weight. This is one of the essential ingredients in order to apply Theorem 1.6.2.

The other essential ingredient is the establishement of a correspondence between the representations of the homogeneous C*-algebraic reduced dual and certain corepresentations of the quantum group admitting non-trivial vectors that are invariant under the action of the subgroup. This is an analogue of Kustermans' result Theorem 1.5.9. For the precise statement, we refer to Theorem 3.7.1. Note that this establishes an analogue of the other essential ingredient of Theorem 1.6.1. The result is worked out in Sections 3.5 - 3.7.

We conclude the chapter by showing that we have collected enough results to show that our structure precisely fits in a Theorem 1.6.2. We discuss how this gives us a Plancherel-Godement theorem. Its proof is a careful modification of the proof of Theorem 1.6.1. In addition, we show how to prove that under certain conditions, a restriction of the Plancherel measure and a certain restriction of the corresponding Duflo-Moore operators gives us the same spherical Plancherel transform. We will treat an example in Chapter 4.

Notation 3.0.3. In this chapter, we fix a locally compact quantum group (M,Δ) together with a closed quantum subgroup (M_1,Δ_1) which we assume to be the compact. Recall [85, Definition 2.9] that this means that we have a surjective $*$-homomorphism $\pi : M_u \to (M_1)_u$ on the level of universal C*-algebras and the induced dual $*$-homomorphism $\hat{\pi} : (\hat{M}_1)_u \to \hat{M}_u$ lifts to a map on the level of von Neumann algebras $\hat{\pi} : \hat{M}_1 \to \hat{M}$, which with slight abuse of notation is denoted by $\hat{\pi}$ again. When we encounter $\hat{\pi}$ in this chapter, we always mean the von Neumann algebraic map.

Note π and $\hat{\pi}$ are in principle also used for the GNS-representations of M and $\hat{M}$. However, we omit the maps most of the time, since M and $\hat{M}$ are identified with their GNS-representations. In that case we explicitly need the GNS-representations, we mention this.

We mention that from a certain point, see Notation 3.1.10, we assume that (M,Δ) is a unimodular quantum group.

We use $\Sigma_{M_1,M} : M_1 \otimes M \to M \otimes M_1$ to denote the flip. The objects associated with (M_1,Δ_1) will be equipped with a subscript 1, i.e. $S_1, R_1, \tau_1, \nu_1, \varphi_1, \ldots$

3.1 Homogeneous spaces and conditional expectations

In this section, we collect some preliminary results on homogeneous spaces as well as conditional expectation values on these spaces. Due to [84, Proposition 3.1], there are canonical right coactions of (M_1, Δ_1) on M, denoted by $\beta, \gamma : M \to M \otimes M_1$, which are normal $*$-homomorphisms uniquely determined by

$$\begin{aligned}(\beta \otimes \iota)(W) &= W_{13}\left((\iota \otimes \hat{\pi})(W_1)\right)_{23},\\ (\gamma \otimes \iota)(W) &= \left((R_1 \otimes \hat{\pi})(W_1)\right)_{23} W_{13}.\end{aligned} \tag{3.1}$$

We have the relation $\gamma = (R \otimes \iota)\beta R$. The map β corresponds classically to right translation, whereas γ corresponds to left translation.

Remark 3.1.1. In [84, Section 3], the roles of (M_1, Δ_1) and $(\hat{M}_1, \hat{\Delta}_1)$ are interchanged. Using our conventions for the roles of (M_1, Δ_1) and $(\hat{M}_1, \hat{\Delta}_1)$, recall the left action μ of (M_1, Δ_1) on M and the left action θ of $(M_1, \Delta^{\rm op})$ on M from [84, Proposition 3.1]. By this proposition, β equals the right coaction $\Sigma_{M_1,M}\theta$ and γ equals the right coaction $\Sigma_{M_1,M}(R_1 \otimes \iota)\mu$.

Definition 3.1.2. We denote M^β for the fixed point algebra

$$M^\beta = \{x \in M \mid \beta(x) = x \otimes 1\}.$$

Similarly, M^γ denotes the fixed point algebra of γ. By definition of γ we find $M^\gamma = R(M^\beta)$. We define

$$N = M^\beta \cap M^\gamma.$$

Note that M^β, M^γ and N are von Neumann algebras. Furthermore, $\Delta(M^\beta) \subseteq M \otimes M^\beta$. Also, by $M^\gamma = R(M^\beta)$, (1.3) and (3.2) below it follows that $\Delta(M^\gamma) \subseteq M^\gamma \otimes M$.

Definition 3.1.3. We recall from [83] that we have normal, faithful operator valued weights,

$$T_\beta : M^+ \to (M^\beta)^+ : x \mapsto (\iota \otimes \varphi_1)\beta(x); \quad T_\gamma : M^+ \to (M^\gamma)^+ : x \mapsto (\iota \otimes \varphi_1)\gamma(x).$$

Since (M_1, Δ_1) is compact, T_β and T_γ are finite. We extend the domains of T_β and T_γ to M in the usual way. We denote the extensions again by T_β and T_γ.

The composition of T_β and T_γ forms a well-defined map on M. Note that $T_\beta(x^*) = T_\beta(x)^*$ and $T_\gamma(x^*) = T_\gamma(x)^*$, where $x \in M$.

Remark 3.1.4. The spaces M^γ, M^β where already introduced in [81] as homogeneous spaces. They also fall within the definition of a homogeneous space as introduced by Kasprzak [46, Remark 3.3]. Moreover, we stretch that T_β and T_γ are conditional expectation values, which properties have been studied in the related papers [72] and [80].

We end this section with the necessary technical preparations.

Lemma 3.1.5. *As maps $M \to M \otimes M \otimes M_1$, we have an equality*

$$(\iota \otimes \Sigma_{M_1,M})(\beta \otimes \iota)\Delta = (\iota \otimes \iota \otimes R_1)(\iota \otimes \gamma)\Delta. \tag{3.2}$$

Proof. For the left hand side, using the pentagon equation and (3.1),

$$(\beta \otimes \iota \otimes \iota)(\Delta \otimes \iota)W = (\beta \otimes \iota \otimes \iota)W_{13}W_{23} = W_{14}((\iota \otimes \hat{\pi})(W_1))_{24}W_{34}.$$

For the right hand side, using again (3.1),

$$\begin{aligned}&(\iota \otimes \iota \otimes R_1 \otimes \iota)(\iota \otimes \gamma \otimes \iota)(\Delta \otimes \iota)W = (\iota \otimes \iota \otimes R_1 \otimes \iota)(\iota \otimes \gamma \otimes \iota)W_{13}W_{23}\\ =&(\iota \otimes \iota \otimes R_1 \otimes \iota)W_{14}((R_1 \otimes \iota)(\iota \otimes \hat{\pi})(W_1))_{34}W_{24} = W_{14}((\iota \otimes \hat{\pi})(W_1))_{34}W_{24}.\end{aligned}$$

The lemma follows by the fact that the elements $\{(\iota \otimes \omega)(W) \mid \omega \in \hat{M}_*\}$ are σ-strong-$*$ dense in M. □

Lemma 3.1.6. *$T_\gamma : M \to M^\gamma$ and $T_\beta : M \to M^\beta$ satisfy the following properties:*

1. *$T_\beta T_\gamma = T_\gamma T_\beta$;*
2. *$\Delta T_\beta = (\iota \otimes T_\beta)\Delta$ and $\Delta T_\gamma = (T_\gamma \otimes \iota)\Delta$;*
3. *$(\iota \otimes T_\gamma)\Delta = (T_\beta \otimes \iota)\Delta$;*
4. *$T_\gamma S \subseteq ST_\beta$ and $T_\beta S \subseteq ST_\gamma$.*

Proof. (1) This follows from the fact $(\iota \otimes \Sigma_{M_1,M_1})(\gamma \otimes \iota)\beta = (\beta \otimes \iota)\gamma$, which can be established as in the proof of Lemma 3.1.5.

(2) We prove that $\Delta T_\gamma = (T_\gamma \otimes \iota)\Delta$, the other equation can be proved similarly using $\beta = (R \otimes \iota)\gamma R$ and (1.3). We find:

$$\begin{aligned}&(\Delta T_\gamma \otimes \iota)(R \otimes \iota)(W)\\ =&(\iota \otimes \iota \otimes \varphi_1 \otimes \iota)(\Delta \otimes \iota \otimes \iota)(R \otimes \iota \otimes \iota)(\beta \otimes \iota)(W)\\ =&(\Sigma_{M,M} \otimes \iota)(R \otimes R \otimes \iota)(\Delta \otimes \iota)(W(1 \otimes \hat{\pi}((\varphi_1 \otimes \iota)(W_1))))\\ =&(R \otimes R \otimes \iota)W_{23}W_{13}(1 \otimes 1 \otimes \hat{\pi}((\varphi_1 \otimes \iota)(W_1)))\\ =&(\iota \otimes \varphi_1 \otimes \iota \otimes \iota)(R \otimes \iota \otimes \iota \otimes \iota)(\beta \otimes R \otimes \iota)W_{23}W_{13}\\ =&(T_\gamma \otimes \iota \otimes \iota)(\Sigma_{M,M} \otimes \iota)(R \otimes R \otimes \iota)W_{13}W_{23}\\ =&(T_\gamma \otimes \iota \otimes \iota)(\Delta \otimes \iota)(R \otimes \iota)(W).\end{aligned}$$

Now, the equation follows by taking slices of the second leg of W.

(3) Since (M_1, Δ_1) is compact, it is unimodular and hence $\varphi_1 = \varphi_1 R_1$. Using Lemma 3.1.5 we find:

$$\begin{aligned}&(\iota \otimes T_\gamma)\Delta = (\iota \otimes \iota \otimes \varphi_1)(\iota \otimes \gamma)\Delta = (\iota \otimes \iota \otimes \varphi_1 R_1)(\iota \otimes \gamma)\Delta\\ =&(\iota \otimes \iota \otimes \varphi_1)\Sigma_{23}(\beta \otimes \iota)\Delta = (T_\beta \otimes \iota)\Delta.\end{aligned}$$

(4) It follows from Lemma 1.2.2 that $(\tau_t \otimes \iota)(W) = (\iota \otimes \hat{\tau}_{-t})(W), t \in \mathbb{R}$. Therefore,

$$(\beta\tau_t \otimes \iota)(W) = (\beta \otimes \hat{\tau}_{-t})(W) = (\iota \otimes \iota \otimes \hat{\tau}_{-t})(W_{13}((\iota \otimes \hat{\pi})(W_1))_{23}).$$

By [59, Proposition 5.45], we know that $\hat{\pi}(\hat{\tau}_1)_t = \hat{\tau}_t\hat{\pi}$. Here $(\hat{\tau}_1)_t$ denotes the scaling group of $(\hat{M}_1, \hat{\Delta}_1)$. Continuing the equation, we find

$$\begin{aligned}(\beta\tau_t \otimes \iota)(W) &= (\iota \otimes \hat{\tau}_{-t})(W)_{13}((\iota \otimes \hat{\pi}(\hat{\tau}_1)_{-t})(W_1))_{23}\\ &=(\tau_t \otimes \iota)(W)_{13}((\tau_1)_t \otimes \hat{\pi})(W_1))_{23} = (\tau_t \otimes (\tau_1)_t \otimes \iota)(\beta \otimes \iota)(W).\end{aligned}$$

We have $\varphi_1(\tau_1)_t = \varphi_1$, since (M_1, Δ_1) is compact. Hence, $T_\beta\tau_t = \tau_t T_\beta$. With a similar computation, involving the relation $(R \otimes \hat{R})(W) = W^*$, see Lemma 1.2.2, we find $T_\beta R = RT_\gamma$. The proposition follows, since $S = R\tau_{-i/2}$.

□

The following proposition is proved in [84].

Proposition 3.1.7 (Proposition 3.1 of [84]). *If $x \in \mathfrak{n}_\varphi$, then $T_\gamma(x) \in \mathfrak{n}_\varphi$ and $\Lambda(T_\gamma(x)) = \hat{\pi}((\varphi_1 \otimes \iota)(W_1))\Lambda(x)$.*

Proposition 3.1.7 defines an orthogonal projection $\hat{\pi}((\varphi_1 \otimes \iota)(W_1))$ for which we simply write

$$P_\gamma = \hat{\pi}((\varphi_1 \otimes \iota)(W_1)).$$

Classically, it corresponds to projecting onto the space of functions that are left invariant with respect to the compact subgroup. Note that $P_\gamma \in \hat{M}$ and

$$N = T_\beta T_\gamma(M) = \overline{\left\{(\iota \otimes \omega_{P_\gamma v, P_\gamma w})(W) \mid v, w \in \mathcal{H}\right\}}^{\sigma-\text{strong}-*}.$$

We need a similar result as Proposition 3.1.7 for T_β. For this we must assert unimodularity of (M, Δ).

First, we prove the following lemma. Note that for $a, b \in \mathcal{T}_\varphi$, we have $a\varphi b \in M_*$ and hence $\mathcal{T}_\varphi\varphi\mathcal{T}_\varphi$ is a subset of M_*. Recall that for $\omega \in \mathcal{I}$, $\xi(\omega) \in \mathcal{H}$ is defined using the Riesz theorem as the unique vector such that $\langle\xi(\omega), \Lambda(x)\rangle = \omega(x^*), x \in \mathfrak{n}_\varphi$. By Lemma 1.2.1, $\mathcal{T}_\varphi\varphi\mathcal{T}_\varphi$ is included in $\mathcal{I}$.

Lemma 3.1.8. *Let $\omega \in \mathcal{I}$. There exists a net $(\omega_j)_j$ in $\mathcal{T}_\varphi\varphi\mathcal{T}_\varphi$ such that $\omega_j \to \omega$ in norm and $\xi(\omega_j) \to \xi(\omega)$ in norm.*

Proof. For $\omega \in \mathcal{I}$ we define the norm $\|\omega\|_{\mathcal{I}} = \max\{\|\omega\|, \|\xi(\omega)\|\}$. We have to prove that $\mathcal{T}_\varphi\varphi\mathcal{T}_\varphi$ is dense in $\mathcal{I}$ with respect to this norm. This is exactly what is obtained Proposition 5.5.4. Indeed, let L and k be as in Chapter 5. As indicated in Section 5.2, $\mathcal{T}_\varphi^2 \subseteq L$ and for $a, b \in \mathcal{T}_\varphi$, $k(ab) = \sigma_i(b)\varphi a$, see also Corollary 5.3.2. So $k(\mathcal{T}_\varphi^2) = \mathcal{T}_\varphi\varphi\mathcal{T}_\varphi$. The proposition yields that $k(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{I}$. □

Proposition 3.1.9. *Suppose that (M, Δ) is unimodular. For $x \in \mathfrak{n}_\varphi$, we have $T_\beta(x) \in \mathfrak{n}_\varphi$. The map $\Lambda(x) \mapsto \Lambda(T_\beta(x))$ is bounded and it extends continuously to the projection $\hat{J}P_\gamma\hat{J}$.*

Proof. By Proposition 3.1.7, we see that $\Lambda(T_\gamma(x^*)) = \Lambda(T_\gamma(x)^*), x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. Denote T for the closure of the map $\Lambda(x) \mapsto \Lambda(x^*), x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. We see that $P_\gamma \mathcal{H}$ is an invariant subspace for T. Since $T = J\nabla^{1/2}$, we find that $\nabla^{it}, t \in \mathbb{R}$ commutes with P_γ.

By Lemma 1.2.3 we find $\hat{\nabla}^{it} = P^{it} J \delta^{it} J$ and by Pontrjagin duality $\nabla^{it} = \hat{P}^{it} \hat{J} \hat{\delta}^{it} \hat{J}$. Using $\delta = 1$ and the self-duality $\hat{P} = P$, we see that $\hat{\nabla}^{it} = \nabla^{it} \hat{J} \hat{\delta}^{-it} \hat{J}$. Since $\hat{\delta}$ is affiliated with $\hat{M}$ and using the previous paragraph, this shows that $P_\gamma \hat{\nabla}^{it} = \hat{\nabla}^{it} P_\gamma$. Hence $\hat{\sigma}_t(P_\gamma) = P_\gamma$.

Now, let $a, b \in \mathcal{T}_{\hat{\varphi}}$ and put $\omega = a\hat{\varphi}b \in \hat{M}_*$. Then,

$$T_\beta\left((\iota \otimes \omega)(W^*)\right) = (\iota \otimes \omega)\left((1 \otimes P_\gamma)W^*\right).$$

Since $P_\gamma \in \hat{M}$ is invariant under $\hat{\sigma}_t$, it follows from [75, Chapter VIII, Lemma 2.4 (ii) and Lemma 2.5] that $\omega P_\gamma \in \hat{\mathcal{I}}$ and

$$\hat{\xi}(\omega P_\gamma) = \hat{\Lambda}(a\hat{\sigma}_{-i}(b)\hat{\sigma}_{-i}(P_\gamma)) = \hat{J}P_\gamma \hat{J}\hat{\Lambda}(a\hat{\sigma}_{-i}(b)).$$

Now, let $\omega \in \hat{\mathcal{I}}$. We prove the proposition for $x = \hat{\lambda}(\omega) \in \mathfrak{n}_\varphi$. Let $(\omega_j)_{j\in J}$ be a net in $\mathcal{T}_{\hat{\varphi}}\hat{\varphi}\mathcal{T}_{\hat{\varphi}}$ that converges to ω and such that $\hat{\xi}(\omega_j)$ converges to $\hat{\xi}(\omega)$, c.f. Lemma 3.1.8. Then, $T_\beta(\hat{\lambda}(\omega_j)) \in \mathfrak{n}_\varphi$ and $T_\beta(\hat{\lambda}(\omega_j)) \to T_\beta(x)$ in the σ-weak topology. Furthermore, $\Lambda(T_\beta(\hat{\lambda}(\omega_j))) = \hat{J}P_\gamma\hat{J}\hat{\xi}(\omega_j)$ is norm convergent. Since Λ is σ-weak/weak closed, and $\mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$, this proves that $T_\beta(x) \in \mathfrak{n}_\varphi$ and $\Lambda(T_\beta(x)) = \hat{J}P_\gamma\hat{J}\Lambda(x)$.

Since the elements in $\hat{\lambda}(\hat{\mathcal{I}})$ form a σ-strong-$*$/norm core for Λ, this proves the proposition. □

We write P_β for the projection $\hat{J}P_\gamma\hat{J}$. In particular $P_\beta \in \hat{M}'$. Under the assumption that (M, Δ) is unimodular, we see that P_β projects onto the elements that are right invariant with respect to the closed quantum subgroup (M_1, Δ_1). Since we need this interpretation of P_β, i.e. Proposition 3.1.9, we assume unimodularity from now on.

Notation 3.1.10. From this point we assume that (M, Δ) is unimodular, i.e. $\varphi = \psi$.

3.2 Homogeneous L^1-spaces and convolution

Here, we study the predual of N_* and embed it into the predual of M_*. We prove various properties of this embedding. In particular, we find a convolution product on N_* that is preserved under this embedding.

At this point it is suitable to recall the definition of a Gelfand pair of compact quantum groups first. For the theory of Hopf algebras, we refer to one of the standard books [48], [79]. It was introduced by Koornwinder in [53]. Consider two unital Hopf-algebras H, H_1. Denote Δ for the comultiplication of H, denote φ_1 for the Haar functional on H_1. Suppose that there exists a surjective morphism $\pi : H \to H_1$, so that H_1 is identified as a quantum subgroup of H. Now consider the left and right coactions $\Delta^l = (\pi \otimes \iota)\Delta, \Delta^r = (\iota \otimes \pi)\Delta$. Define

$$H_1 \backslash H = \{h \in H \mid \Delta^l(h) = 1 \otimes h\}, \qquad H/H_1 = \{h \in H \mid \Delta^r(h) = 1 \otimes h\}$$

and set $H_1\backslash H/H_1 = (H_1\backslash H) \cap (H/H_1)$. Classically, $H_1\backslash H/H_1$ corresponds to the algebra of bi-K-invariant elements. Set

$$\tilde{\Delta} = (\iota \otimes \varphi_1\pi \otimes \iota)(\iota \otimes \Delta)\Delta.$$

Now, the following definition characterizes a quantum Gelfand pair. In fact, there are more equivalent definitions. We state the one which is closest to the theory we develop in the present section.

Definition 3.2.1. Let (H, H_1) be as above. (H, H_1) is called a Gelfand pair if $\tilde{\Delta}$ is cocommutative, i.e. $\tilde{\Delta} = \Sigma_{H,H}\tilde{\Delta}$. The pair is called a strict Gelfand pair if moreover $H_1\backslash H/H_1$ is commutative. Here $\Sigma_{H,H}$ denotes the flip.

A pair of compact groups (G, K) is a Gelfand pair if and only if the Hopf-algebra of matrix coefficients of unitary finite dimensional representations form a quantum Gelfand pair (which is automatically strict). Many deformations of classical Gelfand pairs form strict Gelfand pairs in the Hopf-algebraic setting. Examples can be found in for instance [29], [49], [65], [88] and [89].

We proceed in the von Neumann algebra setting by means of the following definitions.

Definition 3.2.2. For $x \in N$, we define $\Delta^\natural : N \to N \otimes N$ by

$$\Delta^\natural(x) = (\iota \otimes T_\gamma)\Delta(x) = (T_\beta \otimes \iota)\Delta(x), \tag{3.3}$$

see Lemma 3.1.6 (3). This is the von Neumann algebraic version of [89, Eqn. (4)].

Recall that $\Delta(N) \subseteq M^\gamma \otimes M^\beta$, so that $\Delta^\natural(N) \subseteq N \otimes N$. Moreover, using (1) - (3) of Lemma 3.1.6, $\Delta^\natural$ is coassociative, i.e.

$$\begin{aligned} (\iota \otimes \Delta^\natural)\Delta^\natural &= (\iota \otimes \iota \otimes T_\gamma)(\iota \otimes \Delta)(\iota \otimes T_\gamma)\Delta = (\iota \otimes T_\beta T_\gamma \otimes \iota)(\iota \otimes \Delta)\Delta \\ &= (\iota \otimes T_\gamma T_\beta \otimes \iota)(\Delta \otimes \iota)\Delta = (\Delta^\natural \otimes \iota)\Delta^\natural. \end{aligned} \tag{3.4}$$

Note that $\Delta^\natural$ is unital, but generally not multiplicative.

Definition 3.2.3. We define a convolution product $*^\natural$ on N_*,

$$\omega_1 *^\natural \omega_2 = (\omega_1 \otimes \omega_2)\Delta^\natural, \qquad \omega_1, \omega_2 \in N_*.$$

This convolution product is associative, since $\Delta^\natural$ is coassociative by (3.4) .

Definition 3.2.4. If $\omega \in N_*$, then we define

$$\tilde{\omega} = \omega T_\gamma T_\beta = \omega T_\beta T_\gamma \in M_*.$$

For $\omega \in M_*$, we put $\tilde{\omega} = (\omega|_N)^\sim$.

Remark 3.2.5. Note that $\Delta^\natural$ is the von Neumann algebraic version of $\tilde{\Delta}$ [89], which was used to define hypergroup structures. See also the remarks at the beginning of this section. Here, we do not focus on hypergroups for two reasons. First of all, we stay mostly at the measurable von Neumann algebraic setting, which does not allow one to directly define the generalized shift operators [89]. Moreover, we do not assume that N is abelian, i.e. what is called a strict Gelfand pair in [89].

Proposition 3.2.6. *The map $N_* \ni \omega \mapsto \tilde{\omega}$ defines a bijective, norm preserving correspondence between N_* and the functionals $\theta \in M_*$ that satisfy the invariance properties:*

1. $(\theta \otimes \iota)\beta(x) = \theta(x)1_{M_1}$ *for all* $x \in M$;
2. $(\theta \otimes \iota)\gamma(x) = \theta(x)1_{M_1}$ *for all* $x \in M$.

Moreover, for $\omega_1, \omega_2 \in N_*$ we have $\left(\omega_1 *^\natural \omega_2\right)^\sim = \tilde{\omega}_1 * \tilde{\omega}_2$.

Proof. Using right invariance of φ_1, for $x \in M$,

$$\begin{aligned}&(T_\beta \otimes \iota)\beta(x) = (\iota \otimes \varphi_1 \otimes \iota)(\beta \otimes \iota)\beta(x)\\ =&(\iota \otimes \varphi_1 \otimes \iota)(\iota \otimes \Delta_1)\beta(x) = (\iota \otimes \varphi_1)\beta(x) \otimes 1_{M_1} = T_\beta(x) \otimes 1_{M_1}.\end{aligned}$$

Similarly, the right invariance of φ_1 implies $(T_\gamma \otimes \iota)\gamma(x) = T_\gamma(x) \otimes 1_{M_1}, x \in M$. Using (1) of Lemma 3.1.6 one easily verifies that for $\omega \in N_*$, $\tilde{\omega}$ satisfies the invariance properties (1) and (2).

Let $\omega \in N_*$. For $x \in N$ we have $\omega(x) = \tilde{\omega}(x)$, so that $N_* \ni \omega \mapsto \tilde{\omega}$ is injective. If $\theta \in M_*$ satisfies (1), then for $x \in M$,

$$\theta T_\beta(x) = \theta(\iota \otimes \varphi_1)\beta(x) = \varphi_1(\theta \otimes \iota)\beta(x) = \theta(x).$$

Similarly, if $\theta \in M_*$ satisfies (2), then $\theta T_\gamma(x) = \theta(x)$. We find that $\theta = (\theta|_N)^\sim$ if θ satisfies (1) and (2). So $N_* \ni \omega \mapsto \tilde{\omega}$ ranges over the normal functionals on M that satisfy the invariance properties (1) and (2).

Using the left invariance of φ_1, it is a straightforward check that $T_\gamma T_\gamma = T_\gamma$. Then, using (1) - (3) of Lemma 3.1.6, we find:

$$\begin{aligned}(\omega_1 *^\natural \omega_2)^\sim &= (\omega_1 \otimes \omega_2)(\iota \otimes T_\gamma)\Delta T_\gamma T_\beta = (\omega_1 \otimes \omega_2)(T_\gamma \otimes T_\gamma T_\beta)\Delta\\ &= (\omega_1 \otimes \omega_2)(T_\gamma \otimes T_\gamma^2 T_\beta)\Delta = (\omega_1 \otimes \omega_2)(T_\gamma T_\beta \otimes T_\gamma T_\beta)\Delta = \tilde{\omega_1} * \tilde{\omega_2}.\end{aligned}$$

□

Proposition 3.2.7. *For $\omega \in M_*^\sharp$, we have $\tilde{\omega} \in M_*^\sharp$ and $(\tilde{\omega})^* = (\omega^*)^\sim$.*

Proof. Using (4) of Lemma 3.1.6, for $x \in \mathrm{Dom}(S)$, we find

$$\overline{\tilde{\omega}}(S(x)) = \overline{\omega(T_\beta T_\gamma(S(x)^*))} = \overline{\omega((T_\beta T_\gamma(S(x)))^*)} = \overline{\omega(S(T_\beta T_\gamma(x))^*)} = (\omega^*)^\sim(x).$$

So $(\omega^*)^\sim \in M_*$ has the property $((\omega^*)^\sim \otimes \iota)(W) = (\tilde{\omega} \otimes \iota)(W)^*$. This proves that $\tilde{\omega} \in M_*^\sharp$ and $\tilde{\omega}^* = (\omega^*)^\sim$. □

3.3 Dual homogeneous spaces

We associate dual structures to N. We define left and right invariant analogues of the dual von Neumann algebraic quantum group and the universal dual C*-algebraic quantum group. These duals can be constructed by means of the multiplicative unitary W associated with (M, Δ). We define

$$N_*^\sharp = \left\{ \omega \in N_* \mid \exists \theta \in N_* : (\tilde{\omega} \otimes \iota)(W)^* = (\tilde{\theta} \otimes \iota)(W) \right\}. \tag{3.5}$$

For $\omega \in N_*^\sharp$, we set $\|\omega\|_* = \max\{\|\omega\|, \|\omega^*\|\}$. Then, $N_*^\sharp$ becomes a Banach-$*$-algebra with respect to this norm. Proposition 3.2.7 shows that $\omega \in N_*^\sharp$ if and only if $\tilde{\omega} \in M_*^\sharp$. Moreover, $\tilde{\omega}$ has the same norm. Note that $N_*^\sharp$ is dense in N_*. Indeed, the restriction map $M_* \to N_* : \omega \mapsto \omega|_N$ is continuous and the image of the subset $M_*^\sharp \subseteq M_*$ is contained in $N_*^\sharp$ by Proposition 3.2.7. The inclusion $M_*^\sharp \subseteq M_*$ is dense, see [60, Lemma 2.5]. Hence $N_*^\sharp$ is dense in N_*. Using this together with Proposition 3.2.6 and 3.2.7, we see that

$$\hat{N} = \overline{\{(\tilde{\omega} \otimes \iota)(W) \mid \omega \in N_*\}}^{\sigma-\mathrm{strong}-*},$$

is a $*$-subalgebra of $\hat{M}$. Since we can conveniently write $(\tilde{\omega} \otimes \iota)(W) = P_\gamma(\omega \otimes \iota)(W)P_\gamma$, by (3.1), we see that $\hat{N}$ is a von Neumann algebra if considered as acting on $P_\gamma\mathcal{H}$, so that P_γ is its unit. In particular $\hat{N} = P_\gamma \hat{M} P_\gamma$.

We define $\hat{N}_c$ to be the norm closure of the set $\{(\tilde{\omega} \otimes \iota)(W) \mid \omega \in N_*\}$. Then $\hat{N}_c$ is a C*-subalgebra of the reduced dual C*-algebra $\hat{M}_c$.

For $\omega \in N_*^\sharp$, we define

$$\|\omega\|_u^\natural = \sup\{\|\pi(\omega)\| \mid \pi \text{ a representation of } N_*^\sharp\}.$$

Note that this defines a norm since the representation $\omega \mapsto (\tilde{\omega} \otimes \iota)(W)$ is injective as follows using the bijective correspondence established in Proposition 3.2.6. Let $\hat{N}_u$ be the completion of $N_*^\sharp$ with respect to $\| \cdot \|_u^\natural$.

Recall the map $\lambda : M_*^\sharp \to \hat{M} : \omega \mapsto (\omega \otimes \iota)(W)$. We set

$$\lambda^\natural : N_*^\sharp \to \hat{N} : \omega \mapsto (\tilde{\omega} \otimes \iota)(W).$$

Note that the image of $\lambda^\natural$ is contained in $\hat{N}_c$ and we will use this implicitly. $\lambda_u : M_*^\sharp \to \hat{M}_u$ is the canonical inclusion and similarly

$$\lambda_u^\natural : N_*^\sharp \to \hat{N}_u$$

denotes the canonical inclusion.

Recall that $\hat{N}_u$ is a C*-algebra with the following universal property: if π is a representation of $N_*^\sharp$ on a Hilbert space, then there is a unique representation ρ of $\hat{N}_u$ such that $\pi = \rho\lambda_u^\natural$. By this universal property, the map $N_*^\sharp \to \hat{M}_u : \omega \mapsto \lambda_u(\tilde{\omega})$ extends to a representation

$$\iota_u : \hat{N}_u \to \hat{M}_u.$$

Similarly, the map $\lambda^\natural : N_*^\sharp \to \hat{N}_c$ gives rise to a surjective map

$$\hat{\vartheta}^\natural : \hat{N}_u \to \hat{N}_c.$$

In particular, $\hat{\vartheta}\iota_u = \hat{\vartheta}^\natural$, where $\hat{\vartheta} : \hat{M}_u \to \hat{M}_c$ was the canoncial projection induced by the representation $\lambda : M_*^\sharp \to \hat{M}_c$.

Remark 3.3.1. Note that we do not claim that $\iota_u : \hat{N}_u \to \hat{M}_u$ is injective. In fact, this is not true in general, see the comments made in Remark 3.7.5 and the paragraph before this remark.

3.4 Weights on homogeneous spaces

We define weights on the von Neumann algebras and C*-algebras that were introduced in Section 3.1 and 3.3. We study their GNS-representations and prove Proposition 3.4.7, which is essential for implementing Theorem 1.6.2.

Recall that the C*-algebraic weights $\hat{\varphi}_c, \hat{\varphi}_u$ were defined in Section 1.4. The weights on the von Neumann algebras M, $\hat{M}$ and the C*-algebras $\hat{M}_c$, $\hat{M}_u$ restrict to weights on N, $\hat{N}$ and $\hat{N}_c$, $\hat{N}_u$ by setting

$$\varphi^\natural = \varphi|_N, \quad \hat{\varphi}^\natural = \hat{\varphi}|_{\hat{N}} \quad \text{and} \quad \hat{\varphi}_c^\natural = \hat{\varphi}_c|_{\hat{N}_c}, \quad \hat{\varphi}_u^\natural = \hat{\varphi}_u\iota_u = \hat{\varphi}_c\hat{\vartheta}\iota_u = \hat{\varphi}_c^\natural\hat{\vartheta}^\natural,$$

respectively. We prove that $\varphi^\natural$ and $\hat{\varphi}^\natural$ are normal, semi-finite, faithful weights and $\hat{\varphi}_c^\natural$ and $\hat{\varphi}_u^\natural$ are lower semi-continuous, densely-defined, non-zero weights, see Appendix A.3 for definitions. Here the assumption made in Notation 3.1.10 becomes essential.

Proposition 3.4.1. *$\varphi^\natural$ is a normal, semi-finite, faithful weight on N. Its GNS-representation is given by $(P_\gamma P_\beta \mathcal{H}, \Lambda|_{N\cap\mathfrak{n}_\varphi}, \pi|_N)$.*

Proof. Trivially, $\varphi^\natural$ is normal and faithful. Since $\mathfrak{n}_\varphi$ is σ-weakly dense in M, T_β and T_γ are σ-weakly continuous and $N = T_\gamma T_\beta(M)$, Propositions 3.1.7 and 3.1.9 prove that $\varphi^\natural$ is semi-finite. It is straightforward to check that the triple $(P_\gamma P_\beta \mathcal{H}, \Lambda|_{N\cap\mathfrak{n}_\varphi}, \pi|_N)$ satisfies all the properties of a GNS-representation [75, Section VII.1]. □

We define:

$$\mathcal{I}_N = \{\omega \in N_* \mid \tilde{\omega} \in \mathcal{I}\}.$$

Proposition 3.4.2. *For $\omega \in \mathcal{I}$, we find $\tilde{\omega} \in \mathcal{I}$ and $\xi(\tilde{\omega}) = P_\beta P_\gamma \xi(\omega)$. The set $\mathcal{I}_N$ is dense in N_*. The set $\{\xi(\tilde{\omega}) \mid \omega \in \mathcal{I}_N\}$ is a dense subset of $P_\beta P_\gamma \mathcal{H}$.*

Proof. For $x \in \mathfrak{n}_\varphi$, using Propositions 3.1.7 and 3.1.9,

$$\begin{aligned}\langle \xi(\tilde{\omega}), \Lambda(x)\rangle &= \omega(T_\beta T_\gamma(x^*)) = \omega((T_\beta T_\gamma(x))^*)\\ =&\langle \xi(\omega), \Lambda(T_\beta T_\gamma(x))\rangle = \langle P_\beta P_\gamma \xi(\omega), \Lambda(x)\rangle.\end{aligned}$$

The first claim now follows by the definitions of $\mathcal{I}$ and $\xi(\cdot)$. Moreover, we find $\mathcal{I}_N = \{\omega|_N \mid \omega \in \mathcal{I}\}$, so that the second claim follows by Lemma 1.2.1. The last claim also follows from Lemma 1.2.1. □

Proposition 3.4.3. *$\hat{\varphi}^\natural$ is a normal, semi-finite, faithful weight on $\hat{N}$. Its GNS-representation is given by $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}, \hat{\pi}|_{\hat{N}})$.*

Proof. By Proposition 3.4.2, $\{(\tilde{\omega} \otimes \iota)(W) \mid \omega \in \mathcal{I}_N\} \subseteq \hat{N}$ is a σ-strong-$*$ dense subset of $\hat{N}$ contained in $\mathfrak{n}_{\hat{\varphi}}$. This proves that $\hat{\varphi}^\natural$ is semi-finite. Trivially, $\hat{\varphi}^\natural$ is normal and faithful.

To prove the claim about the GNS-representation, we only need to show that the image of $\hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}$ is contained in $P_\beta P_\gamma \mathcal{H}$ and that the inclusion is dense. For $\omega \in \mathcal{I}_N$, we have $\lambda(\tilde{\omega}) \in \hat{N} \cap \mathfrak{n}_{\hat{\varphi}}$ and $\hat{\Lambda}(\lambda(\tilde{\omega})) = \xi(\tilde{\omega}) \in P_\beta P_\gamma \mathcal{H}$ by Proposition 3.4.2. Now, let $x \in \hat{N} \cap \mathfrak{n}_{\hat{\varphi}}$. Since the elements $\lambda(\mathcal{I})$ form a σ-strong-$*$/norm core for $\hat{\Lambda}$, we can find a net $(\omega_j)_{j\in J}$ in $\mathcal{I}$ such that $\lambda(\omega_j) \to x$ in the σ-strong-$*$ topology and $\xi(\omega_j) \to \hat{\Lambda}(x)$ in norm. Consider the net $(\tilde{\omega}_j)_{j\in J}$. We find $\lambda(\tilde{\omega}_j) = P_\gamma \lambda(\omega_j) P_\gamma \to P_\gamma x P_\gamma = x$. And $\xi(\tilde{\omega}_j) = P_\beta P_\gamma \xi(\omega_j)$ is norm convergent to $P_\beta P_\gamma \hat{\Lambda}(x)$. Since $\hat{\Lambda}$ is σ-strong-$*$/norm closed, it follows that $\hat{\Lambda}(x) = P_\beta P_\gamma \hat{\Lambda}(x) \in P_\beta P_\gamma \mathcal{H}$. Moreover, it follows from Proposition 3.4.2 that the range of $\hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}$ is dense in $P_\beta P_\gamma \mathcal{H}$. □

We refer to Definition A.3.1 for the definition of a GNS-representation for a C*-algebraic weight.

Proposition 3.4.4. *$\hat{\varphi}_c^\natural$ is a proper (i.e. densely defined, lower semi-continuous, non-zero) weight on $\hat{N}_c$. Its GNS-representation is $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}|_{\hat{N}_c\cap\mathfrak{n}_{\hat{\varphi}_c}}, \hat{\pi}|_{\hat{N}_c})$.*

Proof. By Proposition 3.4.2, $\{(\tilde{\omega} \otimes \iota)(W) \mid \omega \in \mathcal{I}_N\} \subseteq \hat{N}$ is a norm dense subset of $\hat{N}_c$ contained in $\mathfrak{n}_{\hat{\varphi}}$. The lower semi-continuity and non-triviality follow since $\hat{\varphi}_c^\natural$ is a restriction of the faithful weight $\hat{\varphi}_c$. The claim on the GNS-representation follows exactly as in the proof of Proposition 3.4.3. □

We recall the following lemma from [56].

Lemma 3.4.5 (Lemma 4.2 of [56])**.** *$\mathcal{I} \cap M_*^\sharp$ is dense in $M_*^\sharp$ with respect to the norm $\|\cdot\|_*$.*

Proposition 3.4.6. *$\hat{\varphi}_u^\natural$ is a proper (i.e. densely defined, lower semi-continuous, non-zero) weight on $\hat{N}_u$. Its GNS-representation is given by $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}_u \iota_u, \hat{\pi}_u \iota_u)$.*

Proof. Lemma 3.4.5 shows that $\mathcal{I} \cap M_*^\sharp$ is dense in $M_*^\sharp$. Since $\mathcal{I}_N$ consists of the restrictions to N of functionals in $\mathcal{I}$, see Proposition 3.4.2, and $N_*^\sharp$ consists of the restrictions to N of funtionals in $M_*^\sharp$, see Proposition 3.2.7, it follows that $\mathcal{I}_N \cap N_*^\sharp$ is dense in $N_*^\sharp$. Hence, $\lambda_u^\natural(\mathcal{I}_N \cap N_*^\sharp)$ is dense in $\hat{N}_u$. Moreover, for $\omega \in \mathcal{I}_N \cap N_*^\sharp$,

$$\begin{aligned}&\hat{\varphi}_u^\natural(\lambda_u^\natural(\omega)^*\lambda_u^\natural(\omega)) = \hat{\varphi}_c\hat{\vartheta}\iota_u(\lambda_u^\natural(\omega)^*\lambda_u^\natural(\omega))\\ =&\hat{\varphi}_c\hat{\vartheta}(\lambda_u(\tilde{\omega})^*\lambda_u(\tilde{\omega})) = \hat{\varphi}((\tilde{\omega}\otimes\iota)(W)^*(\tilde{\omega}\otimes\iota)(W)) < \infty.\end{aligned}$$

So $\lambda_u^\natural(\mathcal{I}_N \cap N_*^\sharp)$ is contained in $\mathfrak{n}_{\hat{\varphi}_u^\natural}$. Thus, $\hat{\varphi}_u^\natural$ is densely defined. That $\hat{\varphi}_u^\natural$ is lower semi-continuous is straightforward from the definition, see Section A.3. Take any $\omega \in N_*^\sharp$ such that $\tilde{\omega} \neq 0$, which exists since all functionals in N_* are given by restrictions of functionals in M_*, see Proposition 3.2.6. Then, $\hat{\varphi}_u^\natural(\lambda_u^\natural(\omega^* *^\natural \omega)) = \hat{\varphi}_u(\lambda_u(\tilde{\omega}^* * \tilde{\omega})) = \hat{\varphi}_c(\lambda(\tilde{\omega}^* * \tilde{\omega})) \neq 0$, where $\hat{\varphi}_c$ is the faithful left invariant weight on the reduced C*-algebraic dual $(\hat{M}_c, \hat{\Delta}_c)$.

Finally, we have to prove that $\hat{\Lambda}_u\iota_u$ maps $\hat{N}_u$ densely into $P_\beta P_\gamma \mathcal{H}$. The proof is similar to the one of Proposition 3.4.3, but since the difference is subtle we state it here. Observe that $\lambda(\mathcal{I} \cap M_*^\sharp)$ is a σ-strong-$*$/norm core for $\hat{\Lambda}$ as follows from [60, Proposition 2.6]. So for every $x \in \mathfrak{n}_{\hat{\varphi}_u^\natural}$ we have $(\hat{\vartheta}\iota_u)(x) \in \mathfrak{n}_{\hat{\varphi}}$ and hence there exists a net $(\omega_j)_{j\in J}$ in $\mathcal{I} \cap M_*^\sharp$ such that $\lambda(\omega_j) \to (\hat{\vartheta}\iota_u)(x)$ in the σ-strong-$*$ topology and $\xi(\omega_j) \to \Lambda((\hat{\vartheta}\iota_u)(x)) = \hat{\Lambda}_u(\iota_u(x))$. Consider $(\tilde{\omega}_j)_{j\in J}$. Then, $\lambda(\tilde{\omega_j}) \to (\hat{\vartheta}\iota_u)(x)$ and $\xi(\tilde{\omega}_j) = P_\beta P_\gamma \xi(\omega_j) \to \hat{\Lambda}_u(\iota_u(x))$. Hence $\hat{\Lambda}_u(\iota_u(x)) \in P_\beta P_\gamma \mathcal{H}$. The range of $\hat{\Lambda}_u\iota_u$ is dense in $P_\beta P_\gamma \mathcal{H}$ by Proposition 3.4.2. $\square$

Next, we like to prove that $\hat{\varphi}^\natural$ is essentially the W*-lift of $\hat{\varphi}_u^\natural$. A priori this question is ill-defined, since these weights are defined on different von Neumann algebras. Indeed, $\hat{\varphi}^\natural$ is a weight on $\hat{N}$, which by definition acts on $P_\gamma\mathcal{H}$, whereas the W*-lift of $\hat{\varphi}_u^\natural$ is a weight on $(\hat{\pi}_u\iota_u(\hat{N}_u))''$, which acts on a different Hilbert space. Indeed, since $P_\beta \in \hat{M}'$, we see that $P_\gamma P_\beta \mathcal{H}$ is an invariant subspace of $\hat{N}$. By Proposition 3.4.6, the von Neumann algebra $(\hat{\pi}_u\iota_u(\hat{N}_u))''$ equals the restriction of $\hat{N}$ to $P_\beta P_\gamma \mathcal{H}$, i.e. $(\hat{\pi}_u\iota_u(\hat{N}_u))'' = \hat{N}P_\beta$ acting on $P_\beta P_\gamma \mathcal{H}$.

The point is that that $\hat{N}$ and $\hat{N}P_\beta$ are isomorphic. This follows from Proposition 3.4.3, but we give a different argument here. We claim, more precisely, that the map $\hat{N} \to \hat{N}P_\beta : x \mapsto xP_\beta$ is an isomorphism. Indeed, for any x in the center of $\hat{M}$, $\hat{J}x\hat{J} = x$. Since $P_\beta = \hat{J}P_\gamma\hat{J}$, every projection in the center of $\hat{M}$ majorizes P_β if and only if it majorizes P_γ. Therefore, the central supports of P_β and P_γ are equal. It follows from [42, Theorem 10.3.3] that $\hat{N}$ is isomorphic to $P_\beta\hat{N}P_\beta = \hat{N}P_\beta$, where the isomorphism is given by the map $\hat{N} \to \hat{N}P_\beta : x \mapsto xP_\beta$.

We emphasize that $\hat{N}$ is always considered as a von Neumann algebra acting on $P_\gamma\mathcal{H}$, whereas if we encounter $(\hat{\pi}_u\iota_u(\hat{N}_u))''$ we assume that it acts on $P_\beta P_\gamma\mathcal{H}$.

The described isomorphism $\hat{N} \simeq (\hat{\pi}_u \iota_u(\hat{N}_u))''$, makes the following proposition well-defined. A similar argument holds on the reduced level.

Proposition 3.4.7. *The W*-lifts of $\hat{\varphi}_c^\natural$ and $\hat{\varphi}_u^\natural$ to $\hat{N}$ both equal $\hat{\varphi}^\natural$.*

Proof. We prove the proposition for $\hat{\varphi}_u^\natural$, the proof for $\hat{\varphi}_c^\natural$ is similar. Recall from Proposition 3.4.3 that $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}, \hat{\pi}|_{\hat{N}})$ gives the GNS-representation of $\hat{\varphi}^\natural$. Denote the W*-lift of $\hat{\varphi}_u^\natural$ by ϕ. We denote its GNS-representation by $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$. Recall from Proposition 3.4.6 that the GNS-representation of $\hat{\varphi}_u^\natural$ was given by $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}_u \iota_u, \hat{\pi}_u \iota_u)$.

From Definition A.3.2, the elements $\hat{\pi}_u \iota_u(x), x \in \mathfrak{n}_{\hat{\varphi}_u^\natural}$, form a σ-strong-$*$/norm core for Λ_ϕ and we may take $\mathcal{H}_\phi = P_\gamma P_\beta \mathcal{H}$. If we can prove that the elements $\hat{\pi}_u \iota_u(x), x \in \mathfrak{n}_{\hat{\varphi}_u^\natural}$, also form a σ-strong-$*$/norm core for $\hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}$, then ϕ and $\hat{\varphi}^\natural$ have identical GNS-representations and hence they are equal. So let $x \in \mathfrak{n}_{\hat{\varphi}} \cap \hat{N}$. By [60, Proposition 2.6], $\lambda(\mathcal{I} \cap M_*^\sharp)$ forms a σ-strongly-$*$/norm core for $\hat{\Lambda}$. So let $(\omega_j)_{j\in J}$ be a net in $\mathcal{I} \cap M_*^\sharp$ such that $\lambda(\omega_j) \to x$ in the σ-strong-$*$ topology and $\xi(\omega_j) \to \hat{\Lambda}(x)$ in norm. Then $\xi(\tilde{\omega}_j) = P_\gamma P_\beta \xi(\omega_j)$ converges in norm. Furthermore,

$$\lambda(\tilde{\omega}_j) = (\tilde{\omega}_j \otimes \iota)(W) = P_\gamma(\omega_j \otimes \iota)(W)P_\gamma = P_\gamma \lambda(\omega_j) P_\gamma \to P_\gamma x P_\gamma = x,$$

where the convergence is in the σ-strong-$*$ topology. Note that we have the equality $\lambda(\tilde{\omega}_j) = (\hat{\pi}_u \iota_u \lambda_u^\natural)(\omega_j|_N)$. All in all, we conclude that we have a net $(y_j)_{j\in J} := ((\lambda_u^\natural)(\omega_j|_N))_{j\in J}$ in $\mathfrak{n}_{\hat{\varphi}_u^\natural}$ such that $\hat{\pi}_u \iota_u(y_j)$ is σ-strong-$*$ convergent to x and $\hat{\Lambda}|_{\hat{N}\cap\mathfrak{n}_{\hat{\varphi}}}(\hat{\pi}_u \iota_u(y_j))$ is convergent. □

Remark 3.4.8. In particular, Proposition 3.4.3 implies that $\hat{\varphi}_c^\natural$ and $\hat{\varphi}_u^\natural$ are approximately KMS-weights.

Remark 3.4.9. Note that on one hand, we have a map $N_* \to \hat{N} : \omega \mapsto \lambda(\tilde{\omega}) = (\tilde{\omega} \otimes \iota)(W)$. On the other hand, we can define a map $\hat{N}_* \to N : \omega \mapsto \hat{\lambda}(P_\gamma \omega P_\gamma) = (\iota \otimes \omega)(1 \otimes P_\gamma)(W^*)(1 \otimes P_\gamma)$. These can be considered as the spherical L^1-Fourier transforms. Since both $\varphi^\natural$ and $\hat{\varphi}^\natural$ are normal, semi-finite, faithful weights, one can proceed as in Chapter 5 to obtain a spherical L^p-Fourier transform which then is a restriction of the L^p-Fourier transform as defined in Theorem 5.6.7.

3.5 Spherical corepresentations

We introduce the necessary terminology for corepresentations admitting vectors invariant under the action of a quantum subgroup. These corepresentations can be considered as spherical corepresentations.

Definition 3.5.1. Let $U \in M \otimes B(\mathcal{H}_U)$ be a corepresentation. Then $v \in \mathcal{H}_U$ is called a M_1-invariant vector if

$$(\omega \otimes \iota)((T_\beta \otimes \iota)(U))v = (\omega \otimes \iota)(U)v, \qquad \forall \omega \in M_*.$$

We denote

$$\mathcal{H}_U^{M_1} = \{v \in \mathcal{H}_U \mid v \text{ is } M_1\text{-invariant}\}.$$

Note that $\mathcal{H}_U^{M_1}$ is a closed subspace of $\mathcal{H}_U$. We denote IC(M, M_1) for the equivalence classes of irreducible corepresentations of M that admit non-trivial M_1-invariant vectors. We refer to such corepresentations as *spherical corepresentations*. If $\{(\omega \otimes \iota)(U)\mathcal{H}_U^{M_1} \mid \omega \in M_*\}$ is dense in $\mathcal{H}_U$, then we call U *homogeneously cyclic*. Every irreducible corepresentation of M admitting a non-trivial M_1-invariant vector is homogeneously cyclic

Proposition 3.5.2. *$v \in P_\gamma \mathcal{H}$ if and only if v is M_1-invariant for W.*

Proof. Observe that $(T_\beta \otimes \iota)(W) = W\left(\iota \otimes \hat{\pi}((\varphi_1 \otimes \iota)(W_1))\right) = W(\iota \otimes P_\gamma)$. This yields the only if part. The other implication follows by taking a net $(\omega_j)_{j \in J}$ in M_* such that $\lambda(\omega_j) \to 1$ in the σ-strong-$*$ topology. Then,

$$v \leftarrow (\omega_j \otimes \iota)(W)v = (\omega_j \otimes \iota)(T_\beta \otimes \iota)(W)v = (\omega_j \otimes \iota)(W)P_\gamma v \to P_\gamma v,$$

where the convergence is in norm. □

We mention the following two propositions in order to compare our framework with the setting of classical Gelfand pairs of groups. The proof of the first one is completely analoguous to the proof of [26, Proposition II.6] or [20, Lemma 6.2.3]. For Proposition 3.5.4 one proves that the representation $N_*^\sharp \to B(\mathcal{H}_U^{M_1}) : \omega \to (\tilde{\omega} \otimes \iota)(U)|_{\mathcal{H}_U^{M_1}}$ is irreducible. This can be done along the lines of [26, Théorème III.1] or [20, Proposition 6.31].

Let $U \in M \otimes B(\mathcal{H}_U)$ be a corepresentation. A vector $v \in \mathcal{H}_U$ is called *cyclic* if $\{(\omega \otimes \iota)(U)v \mid \omega \in M_*\}$ is dense in $\mathcal{H}_U$.

Proposition 3.5.3. *Let $U \in M \otimes B(\mathcal{H}_U)$ be a corepresentation. Suppose that U admits a M_1-invariant cyclic vector v. If $\dim(\mathcal{H}_U^{M_1}) = 1$, then U is irreducible.*

Proposition 3.5.4. *Assume that $\hat{N}$ is abelian. Let $U \in M \otimes B(\mathcal{H}_U)$ be an irreducible corepresentation. Then, $\dim(\mathcal{H}_U^{M_1}) \leq 1$.*

In particular, suppose that $\hat{N}$ is abelian, and that $U \in M \otimes B(\mathcal{H}_U)$ is an irreducible corepresentation with M_1-invariant unit vector v. Then $x = (\iota \otimes \omega_{v,v})(U)$ satisfies

$$\Delta^\natural(x) = x \otimes x. \tag{3.6}$$

Hence, x is a character of the convolution algebra $N_*^\sharp$. It can be considered as a quantum spherical function or quantum spherical element. The equality (3.6) allows one to derive product formulae as is done in for example [87], [88], [89]. Here we keep to a more general setting and do not assume that $\hat{N}$ is abelian in order to include the example of $SU_q(1,1)_{\text{ext}}$ in Chapter 4.

3.6 Representations defined by spherical corepresentations

In this section, we show that every spherical corepresentation of M gives rise to a representation of $\hat{N}_u$. We elaborate on some properties that are preserved under this construction. These include direct integration, irreducibility and equivalence. We start with a preliminary lemma.

Lemma 3.6.1. *Let $U \in M \otimes B(\mathcal{H}_U)$ be a corepresentation. Let $v \in \mathcal{H}_U$ and $\omega \in N_*$. The vector $(\tilde{\omega} \otimes \iota)(U)v$ is M_1-invariant.*

Proof. This follows from the following series of equalities for which we use Lemma 3.1.6 (3) and $T_\gamma^2 = T_\gamma$. For $\theta \in M_*$,

$$\begin{aligned}&(\theta \otimes \iota)((T_\beta \otimes \iota)(U))(\tilde{\omega} \otimes \iota)(U)v = (\theta T_\beta \otimes \omega T_\beta T_\gamma \otimes \iota)U_{13}U_{23}v\\ =&(\theta T_\beta \otimes \omega T_\beta T_\gamma \otimes \iota)(\Delta \otimes \iota)(U)v = (\theta \otimes \omega T_\beta T_\gamma \otimes \iota)(\Delta \otimes \iota)(U)v\\ =&(\theta \otimes \iota)(U)(\tilde{\omega} \otimes \iota)(U)v.\end{aligned}$$

□

Remark 3.6.2. Let $U \in M \otimes B(\mathcal{H}_U)$ be a corepresentation. It follows in particular that $\mathcal{H}_U^{M_1}$ is a closed invariant subspace for the representation $N_*^\sharp \to B(\mathcal{H}_U)$: $\omega \mapsto (\tilde{\omega} \otimes \iota)(U)$. By the universal property of $\hat{N}_u$, we see that this gives rise to a representation of $\hat{N}_u$ on $\mathcal{H}_U^{M_1}$. Of course, this representation can be trivial. If U is homogeneously cyclic, then the corresponding representation of $\hat{N}_u$ is non-degenerate. Indeed, suppose that there exist $w \in \mathcal{H}_U^{M_1}$, such that for all $v \in \mathcal{H}_U^{M_1}$ and $\omega \in M_*^\sharp$, $\langle (\tilde{\omega} \otimes \iota)(U)v, w\rangle = 0$. Then, using (4) of Lemma 3.1.6,

$$\begin{aligned}0 =&\langle (\omega T_\gamma T_\beta \otimes \iota)(U)v, w\rangle = \langle (\omega T_\gamma \otimes \iota)(U)v, w\rangle = \langle v, (\omega T_\gamma \otimes \iota)(U)^* w\rangle\\ =&\langle v, (\overline{\omega T_\gamma} \otimes \iota)(U^*)w\rangle = \langle v, (\overline{\omega} T_\gamma \otimes \iota)(S \otimes \iota)(U)w\rangle\\ =&\langle v, (\overline{\omega} \otimes \iota)(S \otimes \iota)(T_\beta \otimes \iota)(U)w\rangle = \langle v, (\omega^* \otimes \iota)(T_\beta \otimes \iota)(U)w\rangle\\ =&\langle v, (\omega^* \otimes \iota)(U)w\rangle = \langle (\omega \otimes \iota)(U)v, w\rangle.\end{aligned}$$

We see that for all $v \in \mathcal{H}_U^{M_1}$ and $\omega \in M_*^\sharp$, $\langle (\omega \otimes \iota)(U)v, w\rangle = 0$, which proves that $w = 0$, since U is homogeneously cyclic. Hence, every non-zero homogeneously cyclic corepresentation U of M gives rise to a non-degenerate representation of $\hat{N}_u$.

Definition 3.6.3. Let $U \in M \otimes B(\mathcal{H}_U)$ be a homogeneously cyclic corepresentation of M on a Hilbert space $\mathcal{H}_U$. Then, we get a representation $\pi_U^\natural$ of $\hat{N}_u$ determined by

$$\pi_U^\natural : \lambda_u^\natural(\omega) \mapsto (\tilde{\omega} \otimes \iota)(U)|_{\mathcal{H}_U^{M_1}}, \qquad \omega \in N_*^\sharp.$$

We emphasize that the representation Hilbert space of $\pi_U^\natural$ is $\mathcal{H}_U^{M_1}$.

Remark 3.6.4. Let $U \in M \otimes B(\mathcal{H}_U)$ be a homogeneously cyclic corepresentation of M. U is irreducible if and only if $\pi_U^\natural$ is irreducible. Indeed, if U is reducible, then clearly $\pi_U^\natural$ is reducible. The only if part follows from a computation similar to the one in Remark 3.6.2.

Recall that we denote π_U for the representation of $\hat{M}_u$ given by $\lambda_u(\omega) \mapsto (\omega \otimes \iota)(U)$. Then, $\pi_U^\natural$ equals the restriction of $\pi_U \iota_u$ to $\mathcal{H}_U^{M_1}$. In the remainder of this section we prove that direct integration as well as irreducibility is preserved under this construction.

We need the following result for Theorem 3.8.1. See Appendix A.1 for the theory of direct integration and the definition of a fundamental sequence.

Proposition 3.6.5. *Let X be a measure space, with standard measure μ. Suppose that for every $x \in X$, we have a homogeneously cyclic corepresentation U_x of M on a Hilbert space $\mathcal{H}_x$ such that $(\pi_{U_x}^\natural)_{x \in X}$ is a μ-measurable field of representations of $\hat{N}_u$. Suppose that $\hat{M}_u$ is separable. Then, $(\mathcal{H}_x)_{x\in X}$ is a μ-measurable field of Hilbert spaces such that $(\mathcal{H}_x^{M_1})_{x\in X}$ is a μ-measurable field of subspaces and $(U_x)_{x\in X}$ is a μ-measurable field of corepresentations.*

Proof. Let $\omega_i \in M_*^\sharp, i \in \mathbb{N}$, be a such that $\lambda_u(\omega_i), i \in \mathbb{N}$, is dense in $\hat{M}_u$. Such ω_i exist because by definition $\hat{M}_u$ is the closure of $\lambda_u(M_*^\sharp)$ and $\hat{M}_u$ is assumed to be separable. Since $(\pi_{U_x}^\natural)_{x\in X}$ is μ-measurable, we have a fundamental sequence $(e_x^j)_{x\in X}, j \in \mathbb{N}$, for the μ-measurable field of Hilbert spaces $(\mathcal{H}_x^{M_1})_{x\in X}$. For $i, j \in \mathbb{N}, x \in X$, define $f_x^{i,j} \in \mathcal{H}_x$ by

$$f_x^{i,j} = (\omega_i \otimes \iota)(U_x)e_x^j.$$

We claim that $(f_x^{i,j})_{x\in X}$ is a fundamental sequence for $(\mathcal{H}_x)_{x\in X}$. Indeed, since U_x is homogeneously cyclic, the span of $(\omega_i \otimes \iota)(U_x)e_x^j, i, j \in \mathbb{N}$ is dense in $\mathcal{H}_x$. Moreover,

$$\begin{aligned}\langle f_x^{i,j}, f_x^{i',j'} \rangle =& \langle (\omega_i \otimes \iota)(U_x)e_x^j, (\omega_{i'} \otimes \iota)(U_x)e_x^{j'} \rangle = \langle ((\omega_{i'}^* * \omega_i) \otimes \iota)(U_x)e_x^j, e_x^{j'} \rangle \\ =& \langle ((\omega_{i'}^* * \omega_i)^\sim \otimes \iota)(U_x)e_x^j, e_x^{j'} \rangle = \langle \pi_{U_x}^\natural(\lambda_u^\natural((\omega_{i'}^* * \omega_i)|_N))e_x^j, e_x^{j'} \rangle,\end{aligned} \tag{3.7}$$

where the third equality follows by a computation similar to the one in Remark 3.6.2. Since $(\pi_{U_x}^\natural)_{x\in X}$ is a μ-measurable field of representations, we see that (3.7) is a μ-measurable function of x. Hence $(f_x^{i,j})_{x\in X}$ is a fundamental sequence. Moreover, by a similar computation as (3.7), for any $\omega \in M_*^\sharp$,

$$\begin{aligned}&\langle (\omega \otimes \iota)(U_x)f_x^{i,j}, f_x^{i',j'} \rangle = \langle ((\omega_{i'}^* * \omega * \omega_i) \otimes \iota)(U_x)e_x^j, e_x^{j'} \rangle \\ =& \langle \pi_{U_x}^\natural(\lambda_u^\natural((\omega_{i'}^* * \omega * \omega_i)|_N))e_x^j, e_x^{j'} \rangle,\end{aligned}$$

is a μ-measurable function of x. For $\omega = \omega_{v,w}$ with $v, w \in \mathrm{Dom}(\hat{\nabla}^{\frac{1}{2}}) \cap \mathrm{Dom}(\hat{\nabla}^{-\frac{1}{2}})$, we have $\omega \in M_*^\sharp$ by [19, Proposition 1.10]. Using [21, Proposition II.1.10 and II.2.1] it is straightforward to prove that $(U_x)_{x\in X}$ is a μ-measurable field of operators. $\square$

Proposition 3.6.6. *Let U_1 and U_2 be homogeneously cyclic corepresentations of M on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Suppose that the representations of $N_*^\sharp$ given by $\pi_i : \omega \mapsto (\tilde{\omega} \otimes \iota)(U_i)|_{\mathcal{H}_{U_i}^{M_1}}, \omega \in N_*^\sharp, i \in \{1, 2\}$ are equivalent. Then U_1 and U_2 are equivalent.*

Proof. Let $T : \mathcal{H}_{U_1}^{M_1} \to \mathcal{H}_{U_2}^{M_1}$ be the unitary intertwiner between π_1 and π_2. Let Q_0 be the mapping

$$\left\{(\omega \otimes \iota)(U_1)v \mid \omega \in M_*^\sharp, v \in \mathcal{H}_{U_1}^{M_1}\right\} \to \left\{(\omega \otimes \iota)(U_2)w \mid \omega \in M_*^\sharp, w \in \mathcal{H}_{U_2}^{M_1}\right\}$$
$$(\omega \otimes \iota)(U_1)v \mapsto (\omega \otimes \iota)(U_2)Tv.$$

This map is well-defined and isometric. Indeed, for $\omega \in M_*^\sharp$ and $v \in \mathcal{H}_{U_1}^{M_1}$,

$$\begin{aligned}&\|(\omega \otimes \iota)(U_2)Tv\|^2 = \langle (\omega^* * \omega \otimes \iota)(U_2)Tv, Tv\rangle = \langle ((\omega^* * \omega)^\sim \otimes \iota)(U_2)Tv, Tv\rangle\\ =&\langle ((\omega^* * \omega)^\sim \otimes \iota)(U_1)v, v\rangle = \langle (\omega^* * \omega \otimes \iota)(U_1)v, v\rangle = \|(\omega \otimes \iota)(U_1)v\|^2,\end{aligned}$$

where the second equality follows from a similar calculation as in Remark 3.6.2. Since U_1 and U_2 are homogeneously cyclic, Q_0 is densely defined and has dense range. Let $Q : \mathcal{H}_{U_1} \to \mathcal{H}_{U_2}$ be the unitary extension of Q_0. Let $\omega, \omega_1, \omega_2 \in M_*^\sharp$ and $v, w \in \mathcal{H}_{U_1}^{M_1}$,

$$\begin{aligned}&\langle (\omega \otimes \iota)(U_1)(\omega_1 \otimes \iota)(U_1)v, (\omega_2 \otimes \iota)(U_1)w\rangle = \langle (\omega_2^* * \omega * \omega_1 \otimes \iota)(U_1)v, w\rangle\\ =&\langle ((\omega_2^* * \omega * \omega_1)^\sim \otimes \iota)(U_1)v, w\rangle = \langle ((\omega_2^* * \omega * \omega_1)^\sim \otimes \iota)(U_2)Tv, Tw\rangle\\ =&\langle (\omega_2^* * \omega * \omega_1 \otimes \iota)(U_2)Tv, Tw\rangle = \langle (\omega \otimes \iota)(U_2)(\omega_1 \otimes \iota)(U_2)Tv, (\omega_2 \otimes \iota)(U_2)Tw\rangle\\ =&\langle (\omega \otimes \iota)(U_2)Q(\omega_1 \otimes \iota)(U_1)v, Q(\omega_2 \otimes \iota)(U_1)w\rangle,\end{aligned}$$

where the second equality follows again from a similar calculation as in Remark 3.6.2. Since U_1 is homogeneously cyclic, this proves that Q intertwines U_1 with U_2. □

Note that the converse of the previous proposition is clear: if U_1 and U_2 are equivalent corepresentations, then the corresponding representations as considered in Remark 3.6.2 are equivalent.

Remark 3.6.7. Proposition 3.6.6 and its converse also hold on the universal level. So let U_1 and U_2 be homogeneously cyclic corepresentations of M. $\pi_{U_1}^\natural$ and $\pi_{U_2}^\natural$ are equivalent if and only if U_1 and U_2 are equivalent.

3.7 Representations of $\hat{M}_u, \hat{M}_c, \hat{N}_u$ and $\hat{N}_c$

In this section we compare the representations of the C*-algebras defined Section 3.3 and the C*-algebras of the reduced and universal dual. The main objective is to prove that the representations of $\hat{N}_c$ 'lift' to representations of $\hat{M}_c$.

Let us give a more elaborate discussion. There are three special types of representations within Rep$(\hat{N}_u)$.

1. As explained in Remark 3.6.2, the corepresentations of M give rise to representations of $\hat{N}_u$. Recall Theorem 1.5.9 that the corepresentations of M are in 1-to-1 correspondence with non-degenerate representations of $\hat{M}_u$. Hence, the representations of $\hat{M}_u$ give rise to representations of $\hat{N}_u$. This correspondence can be described more directly: if π is a representation of $\hat{M}_u$ on a Hilbert space $\mathcal{H}_\pi$, then $\pi\iota_u$ is the corresponding representation of $\hat{N}_u$ on the closure of $((\pi\iota_u)(\hat{N}_u))\mathcal{H}_\pi$. By Remark 3.6.7 this assignment descends to a well-defined map on the equivalence classes of representations of $\hat{M}_u$ that is injective on the representations that correspond to IC(M, M_1),
$$\mathrm{Rep}(\hat{M}_u) \to \mathrm{Rep}(\hat{N}_u) : \pi \mapsto \pi\iota_u.$$

2. If π is a representation of $\hat{N}_c$, then $\pi\hat{\vartheta}^\natural$ is a representation of $\hat{N}_u$. These representations correspond to the representations that are weakly contained in the GNS-representation of $\hat{\varphi}_u^\natural$. Indeed this follows, since by Proposition 3.4.6, this GNS-representation is given by
$$\hat{\pi}_u\iota_u = \hat{\pi}_c\hat{\vartheta}\iota_u = \hat{\pi}_c\hat{\vartheta}^\natural = \hat{\pi}|_{\hat{N}_c}\hat{\vartheta}^\natural.$$
Hence, every representation $\pi\hat{\vartheta}^\natural$, with $\pi \in \mathrm{Rep}(\hat{N}_c)$ is weakly contained in the GNS-representation of $\hat{N}_u$. The other way around, any representation of $\hat{N}_u$ that is weakly contained in the GNS-representation of $\hat{\varphi}_u^\natural$ factors through the canonical projection $\hat{\vartheta}^\natural$.

3. If π is a representation of $\hat{M}_c$, then $\pi\hat{\vartheta}^\natural = \pi\hat{\vartheta}\iota_u$ is a representation of $\hat{N}_u$. Here, we used that $\hat{N}_c \subseteq \hat{M}_c$.

The main result of this section is the following. We prove that every representation of $\hat{N}_c$ comes from a representation of $\hat{M}_c$, i.e. the representations of $\hat{N}_u$ obtained in 2. and 3. are the same ones.

Theorem 3.7.1. *For every non-degenerate representation $\rho \in \mathrm{Rep}(\hat{N}_c)$, there exists a non-degenerate representation $\pi \in \mathrm{Rep}(\hat{M}_c)$ on a Hilbert space $\mathcal{H}_\pi$ such that ρ is equivalent to the restriction of $\pi|_{\hat{N}_c}$ to the closure of $\pi(\hat{N}_c)\mathcal{H}_\pi$.*

Proof. Let $M_*^{\sharp,\beta}$ denote space of functionals $\theta \in M_*^\sharp$ such that $\theta T_\beta = \theta$. By a similar argument as in the proof of Proposition 3.2.7, we see that if $\theta \in M_*^\sharp$, then $\theta T_\beta \in M_*^{\sharp,\beta}$ and $(\theta T_\beta)^* = \theta^* T_\gamma$. In particular, $(N_*^\sharp)^\sim \subseteq M_*^{\sharp,\beta}$. Note that for $\theta_1, \theta_2 \in M_*^{\sharp,\beta}$, we have that $(\theta_1^* * \theta_2)^\sim = \theta_1^* * \theta_2$.

We complete $M_*^{\sharp,\beta}$ into a (right) Hilbert $\hat{N}_c$-module. For $\theta, \theta_1, \theta_2 \in M_*^{\sharp,\beta}$ and $\omega \in N_*^\sharp$, we put
$$\theta \cdot \omega = \theta * \tilde{\omega} \in M_*^{\sharp,\beta} \tag{3.8}$$
$$\langle \theta_1, \theta_2 \rangle_{N_*^\sharp} = (\theta_1^* * \theta_2)|_N \in N_*^\sharp. \tag{3.9}$$

The fact that (3.8) is in $M_*^{\sharp,\beta}$ follows from Lemma 3.1.6. The inner product given by (3.9) indeed has values in $N_*^\sharp$ as follows from Proposition 3.2.7. This gives a right $N_*^\sharp$-module structure on $M_*^{\sharp,\beta}$. We apply [71, Lemma 2.16] to get a Hilbert $\hat{N}_c$-module X. Here we consider $N_*^\sharp$ is a subalgebra of $\hat{N}_c$ by means of the map $\lambda^\natural$. To continue, note that conditions (a) and (c) of [71, Definition 2.1] are indeed satisfied. (b) follows, since for $\theta_1, \theta_2 \in M_*^{\sharp,\beta}$ and $\omega \in N_*^\sharp$,

$$\langle \theta_1, \theta_2 \cdot \omega \rangle_{N_*^\sharp} = (\theta_1^* \otimes \theta_2 \otimes \omega)(\iota \otimes \iota \otimes T_\beta T_\gamma)(\Delta \otimes \iota)\Delta|_N,$$
$$\langle \theta_1, \theta_2 \rangle_{N_*^\sharp} *^\natural \omega = (\theta_1^* \otimes \theta_2)\Delta|_N *^\natural \omega = (\theta_1^* \otimes \theta_2 \otimes \omega)(\iota \otimes \iota \otimes T_\gamma)(\Delta \otimes \iota)\Delta|_N.$$

Since $(\iota \otimes T_\beta)\Delta|_N = \Delta T_\beta|_N = \Delta|_N$, these expressions are equal. (d) follows, since for $\theta \in M_*^{\sharp,\beta}$,

$$\lambda^\natural((\theta^* * \theta)|_N) = \lambda(\theta^* * \theta) = \lambda(\theta)^*\lambda(\theta) \geq 0,$$

in $\hat{N}_c$. This defines the right Hilbert $\hat{N}_c$-module X and we denote its norm by $\|\cdot\|_X$.

We are able to define an action of $M_*^\sharp$ on X by means of adjointable operators which extends the convolution product on $M_*^\sharp$. Indeed, for $\omega \in M_*^\sharp$, and $\theta_1, \theta_2 \in M_*^{\sharp,\beta}$,

$$\langle \omega * \theta_1, \theta_2 \rangle_{N_*^\sharp} = \theta_1^* * \omega^* * \theta_2|_N = \langle \theta_1, \omega^* * \theta_2 \rangle_{N_*^\sharp}. \tag{3.10}$$

Furthermore, for $\omega \in M_*^\sharp$ and $\theta \in M_*^{\sharp,\beta}$,

$$\|\omega * \theta\|_X^2 = \|\lambda(\theta^* * \omega^* * \omega * \theta)\| \leq \|\lambda(\omega)\|^2 \|\lambda(\theta^* * \theta)\| = \|\lambda(\omega)\|^2 \|\theta\|_X^2. \tag{3.11}$$

So the action of $\omega \in M_*^\sharp$ is bounded. This allows us to extend the action of ω to an adjointable operator on X. We denote this operator by L_ω. Moreover, from (3.11) we get a representation of $\hat{M}_c$ on the Hilbert module X by means of adjointable operators. This representation is uniquely determined by $\lambda^\natural(\omega) \mapsto L_\omega, \omega \in N_*^\sharp$.

Now, let ρ be a representation of $\hat{N}_c$ on a Hilbert space $\mathcal{H}_\rho$. By [71, Proposition 2.66] we get an induced representation of $\hat{M}_c$ on $X \otimes_{\hat{N}_c} \mathcal{H}_\rho$. Let us denote the latter by $\mathrm{Ind}\rho$. Let $(a_j)_{j \in J}$ be an approximate unit for $\hat{M}_c$. Then, for $\theta \in M_*^{\sharp,\beta}, v \in \mathcal{H}_\rho$, we see that $(\mathrm{Ind}\rho)(a_j)(\theta \otimes v) \to (\theta \otimes v)$ in X. So $\mathrm{Ind}\rho$ is non-degenerate.

Note that the completion of $(N_*^\sharp)^\sim \otimes_{\hat{N}_c} \mathcal{H}_\rho$ is a closed subspace of the Hilbert space $X \otimes_{\hat{N}_c} \mathcal{H}_\rho$ that is isomorphic to $\mathcal{H}_\rho$ via the unitary extension T of

$$(N_*^\sharp)^\sim \otimes_{\hat{N}_c} \mathcal{H}_\rho \to \mathcal{H}_\rho : \tilde{\omega} \otimes v \mapsto \rho(\lambda^\natural(\omega))v. \tag{3.12}$$

The map (3.12) extends unitarily since ρ is non-degenerate. For $\omega, \theta \in N_*^\sharp$, $v \in \mathcal{H}_\rho$,

$$(\mathrm{Ind}\rho)(\lambda^\natural(\omega))(\tilde{\theta} \otimes v) = \tilde{\omega} * \tilde{\theta} \otimes v = (\omega *^\natural \theta)^\sim \otimes v,$$

so that $(N_*^\sharp)^\sim \otimes_{\hat{N}_c} \mathcal{H}_\rho$ is an invariant subspace for $\mathrm{Ind}\rho$. We denote its closure by Y. Moreover, for $\omega, \theta \in N_*^\sharp$ and $v \in \mathcal{H}_\rho$,

$$\begin{aligned} &T\ (\mathrm{Ind}\rho)(\lambda^\natural(\omega))\ (\tilde{\theta} \otimes v) = T((\tilde{\omega} * \tilde{\theta}) \otimes v) = T((\omega *^\natural \theta)^\sim \otimes v) \\ =&\rho(\lambda^\natural(\omega *^\natural \theta))v = \rho(\lambda^\natural(\omega))\rho(\lambda^\natural(\theta))v = \rho(\lambda^\natural(\omega))T(\tilde{\theta} \otimes v) \end{aligned}$$

so that T intertwines $(\mathrm{Ind}\rho)|_{\hat{N}_c}$ restricted to the Hilbert space Y with ρ. Finally, we claim that Y equals the closure of $(\mathrm{Ind}\rho)(\hat{N}_c)X$. Indeed, for any $\omega \in N_*^\sharp$, $\theta \in M_*^{\sharp,\beta}$ and $v \in \mathcal{H}_\rho$, we see that $(\mathrm{Ind}\rho)(\lambda^\natural(\omega))(\theta \otimes v) = (\tilde{\omega} * \theta) \otimes v = (\omega *^\natural \theta|_N)^\sim \otimes v \in Y$. Since $\hat{N}_c\hat{N}_c \subseteq \hat{N}_c$ is dense, it is straightforward to prove that Y equals the closure of $(\mathrm{Ind}\rho)(\hat{N}_c)X$ in X. This concludes the proof by choosing $\pi = \mathrm{Ind}\rho$. □

Corollary 3.7.2. *For every representation ρ of $\hat{N}_u$ that factors through $\hat{\vartheta}^\natural$, there is a homogeneously cyclic corepresentation U of M such that ρ is equivalent to $\pi_U^\natural$.*

Remark 3.7.3. An essential ingredient for the proof of the quantum version of the Plancherel-Godement theorem is to see to which corepresentation the GNS-map of $\hat{N}_u$ corresponds. Recall that the GNS-representation of $\hat{\varphi}_u^\natural$ was given by the triple $(P_\gamma P_\beta \mathcal{H}, \hat{\Lambda}_u \iota_u, \hat{\pi}_u \iota_u)$, see Proposition 3.4.6. Since $\hat{\pi}_u \iota_u = \hat{\pi}|_{\hat{N}_c} \hat{\vartheta}^\natural$, we can apply Corollary 3.7.2. We define the closed subspace

$$\mathcal{E} = \overline{\{(\tilde{\omega} \otimes \iota)(W) P_\beta P_\gamma \mathcal{H} \mid \omega \in N_*^\sharp\}} = \overline{\{(\hat{\pi}_u \iota_u \lambda_u^\natural(\omega) P_\beta P_\gamma \mathcal{H} \mid \omega \in N_*^\sharp\}} \subseteq \mathcal{H}, \tag{3.13}$$

where the closure is with respect to the norm in $\mathcal{H}$. It is clear that the representation of $\hat{N}_u$ that corresponds to the restriction of W to $\mathcal{E}$ equals $\hat{\pi}_u \iota_u$.

We use the notation $\mathrm{IR}(\hat{M}_u, M_1)$ to denote the irreducible representations π of $\hat{M}_u$ such that the representation $\pi\iota_u$ is non-trivial. Under the 1-1 correspondence between $\mathrm{IR}(\hat{M}_u)$ and $\mathrm{IC}(M)$, see Theorem 1.5.9, we see from Remark 3.6.7 and the remarks following Definition 3.6.3 that $\mathrm{IR}(\hat{M}_u, M_1)$ corresponds to $\mathrm{IC}(M, M_1)$. Let $\mathrm{IR}(\hat{M}_c, M_1)$ denote the irreducible representations of $\hat{M}_c$ such that the restriction to $\hat{N}_c$ is non-trivial. We find the following diagram of inclusions.

$$\begin{tikzcd} & \mathrm{IR}(\hat{N}_u) & \mathrm{IR}(\hat{N}_c) \arrow[l, hook'] \\ \mathrm{IC}(M, M_1) \arrow[r, phantom, "\simeq"] & \mathrm{IR}(\hat{M}_u, M_1) \arrow[u, hook] & \mathrm{IR}(\hat{M}_c, M_1) \arrow[l, hook'] \arrow[u, hook, "\simeq"] \end{tikzcd} \tag{3.14}$$

Note that the map $\mathrm{IR}(\hat{M}_u, M_1) \hookrightarrow \mathrm{IR}(\hat{N}_u)$ indeed maps into the representations of $\hat{N}_u$ that are irreducible, c.f. Remark 3.6.4. Hence, also the vertical inclusion on the right hand side of (3.14) preserves irreducibility.

The example in Chapter 4 shows that the inclusion given by $\mathrm{IR}(\hat{M}_c, M_1) \hookrightarrow \mathrm{IR}(\hat{M}_u, M_1)$ is not surjective. We briefly comment on the fact that also the inclusion given by $\mathrm{IR}(\hat{M}_u, M_1) \hookrightarrow \mathrm{IR}(\hat{N}_u)$ is generally not surjective. This is a consequence of the fact that there are Lie groups G with compact subgroup K for which there are non-unitary representations whose restriction to the bi-K-invariant functions forms a representation, i.e. a homomorphism that preserves the $*$-operation, whereas the representation of all L^1-functions on G does not preserve the $*$. This happens for example for $SL(2, \mathbb{R})$, see [95, Example 1.1.2

on p. 37 and p. 40]. This shows that the induction argument contained in the proof of Theorem 3.7.1 does not work in general on the universal level.

Remark 3.7.4. We strongly believe that also $SU_q(1,1)_{\mathrm{ext}}$ is an example for which $\mathrm{IR}(\hat{M}_c, M_1) \hookrightarrow \mathrm{IR}(\hat{M}_u, M_1)$ is not surjective, since the matrix coefficients of the corepresentations $W_{p,x}, x \in [-1,1], p \in q^{\mathbb{Z}}$, can be extended analytically to a neighbourhood of $[-1,1]$, see [35, Section 10.3]. We expect that a similar phenomenon as described above occurs. However, it would require some work to make this precise.

Remark 3.7.5. Assume the map $\iota_u : \hat{N}_u \to \hat{M}_u$ to be injective. Then by general C*-algebra theory, it is isometric. With this additional assumption Theorem 3.7.1 holds on the universal level. So for every $\rho \in \mathrm{Rep}(\hat{N}_u)$, there is a representation $\pi \in \mathrm{Rep}(\hat{M}_u)$ on a Hilbert space $\mathcal{H}_\pi$, such that ρ is equivalent to the restriction of $\pi\iota_u$ to $(\pi\iota_u(\hat{N}_u))\mathcal{H}_\pi$.

The proof is complelety analogous to the one of Theorem 3.7.1, where one takes the universal norm instead of the reduced norm on $N_*^\sharp$. The injectivity of ι_u plays an essential role at two places. First of all, the injectivity of ι_u can be used to prove positivity of the inner product (3.9), since in this case an element in $\hat{N}_u$ is positive if and only if it is positive in $\hat{M}_u$. Secondly, the universal analogue of (3.11) can be recovered from the injectivity of ι_u, since for $\omega \in M_*^\sharp$ and $\theta \in M_*^{\sharp,\beta}$,

$$
\begin{aligned}
&\|\lambda_u^\natural(\theta^* * \omega^* * \omega * \theta)\|_u^\natural = \|\lambda_u(\theta^* * \omega^* * \omega * \theta)\|_u \\
\leq &\|\lambda_u(\omega^* * \omega)\|_u \|\lambda_u(\theta^* * \theta)\|_u = \|\lambda_u(\omega)\|_u \|\lambda_u^\natural(\theta^* * \theta)\|_u^\natural .
\end{aligned}
$$

The rest of the prove of Theorem 3.7.1 can be copied mutatis mutandis.

3.8 A quantum group analogue of the Plancherel-Godement theorem

Here we prove a decomposition theorem that may be considered as a locally compact quantum group version of the Plancherel-Godement theorem as can be found in [26, Théorème IV.2]. The proof is different from the one given in [26] and follows the line of the Plancherel theorem as proved by Desmedt, c.f. Theorem 1.6.1. We show that the C*-algebra $\hat{N}_u$ together with the weight $\varphi_u^\natural$ that we have introduced and studied so far fit into the framework of Theorem 1.6.2. Then we use Theorem 3.7.1 to translate the results in terms of corepresentations of M that admit a M_1-invariant vector.

Recall that we define the space $\mathcal{E}$ in (3.13). Let $\mathcal{L}$ be any Hilbert space and let $\mathcal{L}_0 \subseteq \mathcal{L}$ be a closed subspace. We denote the conjugate Hilbert space of $\mathcal{L}$ by $\overline{\mathcal{L}}$. Note that the space $\mathcal{L} \otimes \overline{\mathcal{L}_0}$ can canonically be identified with the Hilbert-Schmidt operators in $B(\mathcal{L}_0, \mathcal{L})$. We denote the latter space by $B_2(\mathcal{L}_0, \mathcal{L})$. For results on direct integration we refer to Appendix A.1. In particular, we use Theorem A.1.13 implicitly several times. If A and B are unbounded operators such that AB is closable, we denote $A \cdot B$ for the closure of AB.

Theorem 3.8.1 (Plancherel-Godement). *Let (M, Δ) be a unimodular locally compact quantum group and let (M_1, Δ_1) be a compact (closed) quantum subgroup. Let $\hat{N}$ and $\hat{N}_u$ be the von Neumann algebra and C*-algebra as defined earlier in this section. Suppose that $\hat{N}$ is a type I von Neumann algebra and that $\hat{N}_u$ and $\hat{M}_u$ are separable.*

Then, there exists a standard measure μ^{M_1} on IC(M, M_1), a μ^{M_1}-measurable field of Hilbert spaces $(\mathcal{H}_U)_{U \in IC(M,M_1)}$ of which $(\mathcal{H}_U^{M_1})_{U \in IC(M,M_1)}$ forms a measurable field of subspaces, a measurable field of self-adjoint strictly positive operators $(D_U^{M_1})_{U \in IC(M,M_1)}$ acting on $\mathcal{H}_U^{M_1}$ and an isomorphism $\mathcal{Q}^{M_1}$ of $\mathcal{E}$ onto $\int_{IC(M,M_1)}^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U^{M_1}} d\mu^{M_1}(U)$ with properties:

1. *For $\omega \in \mathcal{I}_N$ and μ^{M_1}-almost all $U \in$ IC(M, M_1), the operator given by $(\tilde{\omega} \otimes \iota)(U)(D_U^{M_1})^{-1}$ is bounded and $(\tilde{\omega} \otimes \iota)(U) \cdot (D_U^{M_1})^{-1}$ is in $B_2(\mathcal{H}_U^{M_1})$.*

2. *For $\omega_1, \omega_2 \in \mathcal{I}_N$, we have*

$$\begin{aligned} &\langle \xi(\tilde{\omega}_1), \xi(\tilde{\omega}_2) \rangle \\ &= \int_{IC(M,M_1)} \mathrm{Tr}\left(((\tilde{\omega}_2 \otimes \iota)(U) \cdot (D_U^{M_1})^{-1})^* ((\tilde{\omega}_1 \otimes \iota)(U) \cdot (D_U^{M_1})^{-1}) \right) d\mu^{M_1}(U), \end{aligned}$$

   and we let $\mathcal{Q}_0^{M_1} : P_\beta P_\gamma \mathcal{H} \to \int_{IC(M,M_1)}^{\oplus} \mathcal{H}_U^{M_1} \otimes \overline{\mathcal{H}_U^{M_1}} d\mu^{M_1}(U)$ be the isometric extension of

$$\begin{aligned} \hat{\Lambda}(\lambda(\tilde{\mathcal{I}}_N)) \to & \int_{IC(M,M_1)}^{\oplus} B_2(\mathcal{H}_U^{M_1}) d\mu(U) : \\ \xi(\tilde{\omega}) \mapsto & \int_{IC(M,M_1)}^{\oplus} (\tilde{\omega} \otimes \iota)(U) \cdot (D_U^{M_1})^{-1} d\mu^{M_1}(U). \end{aligned}$$

3. *$\mathcal{Q}_0^{M_1}$ intertwines $\hat{\pi}_u \iota_u$ and $\int_{IC(M,M_1)}^{\oplus} \pi_U^\natural \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu^{M_1}(U)$.*

4. *$\mathcal{Q}^{M_1}$ intertwines the restriction of W to $\mathcal{E}$ with $\int_{IC(M,M_1)}^{\oplus} U \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu^{M_1}(U)$. Moreover, the restriction of $\mathcal{Q}^{M_1}$ to $P_\beta P_\gamma \mathcal{H} \to \int_{IC(M,M_1)}^{\oplus} \mathcal{H}_U^{M_1} \otimes \overline{\mathcal{H}_U^{M_1}} d\mu^{M_1}(U)$ equals $\mathcal{Q}_0^{M_1}$.*

5. *Assume moreover that $\hat{M}$ is a type I von Neumann algebra. Let μ, D_U, $\mathcal{Q}$ be defined as in Theorem 1.6.1. Then $\mathcal{H}_U^{M_1}$ is an invariant subspace for μ-almost all D_U and IC(M, M_1) is a μ-measurable subset of $\mathrm{IC}(M)$. If one takes:*

   - *μ^{M_1} equal to the restriction of μ to IC(M, M_1);*

- $D_U^{M_1}$ *equal to the restriction of* D_U *to* $\mathcal{H}_U^{M_1}$*;*
- $\mathcal{Q}^{M_1}$ *the restriction of* $\mathcal{Q}$ *to* $\mathcal{E}$*.*

Then, μ^{M_1}*,* $D_U^{M_1}$*,* $\mathcal{Q}^{M_1}$ *satisfy the properties (1) - (4).*

Proof. By Proposition 3.4.6 and Proposition 3.4.7 $\hat{\varphi}_u^\natural$ is a proper approximate KMS-weight. Therefore, we can apply Theorem 1.6.2, so that we obtain a measure μ^{M_1} on $\mathrm{IR}(\hat{N}_u)$, a measurable field of Hilbert spaces $(\mathcal{K}_\sigma^{M_1})_{\sigma\in\mathrm{IR}(\hat{N}_u)}$, a measurable field of representations $(\pi_\sigma)_{\sigma\in\mathrm{IR}(\hat{N}_u)}$, a measurable field of self-adjoint, strictly positive operators $(D_\sigma^{M_1})_{\sigma\in\mathrm{IR}(\hat{N}_u)}$ and an isomorphism $\mathcal{Q}_0^{M_1}$ of $P_\gamma P_\beta \mathcal{H}$ onto $\int_{\mathrm{IR}(\hat{N}_u)}^\oplus \mathcal{K}_\sigma^{M_1} \otimes \overline{\mathcal{K}_\sigma^{M_1}} d\mu^{M_1}(\sigma)$ satisfying the properties of Theorem 1.6.2.

Let $\rho \in \mathrm{IR}(\hat{N}_u)$ be in the support of μ^{M_1}. We claim that π_ρ is weakly contained in $\hat{\pi}_u\iota_u$. Suppose that this is not the case, so that there exists $x \in \hat{N}_u$ such that $\pi_\rho(x) \neq 0$ but $\hat{\pi}_u\iota_u(x) = 0$. Let $X = \{\sigma \in \mathrm{IR}(\hat{N}_u) \mid \pi_\sigma(x) \neq 0\}$. Then X is an open neighbourhood of ρ. Moreover, it follows form Theorem 1.6.2 that $\mathcal{Q}_0^{M_1}$ intertwines $\hat{\pi}_u\iota_u$ with $\int_{\mathrm{IR}(\hat{N}_u)}^\oplus \pi_\sigma \otimes \overline{1_{\mathcal{K}_\sigma}} d\mu^{M_1}(\sigma)$ from which it follows that $\mu^{M_1}(X) = 0$. This contradicts the fact that ρ is in the support of μ^{M_1}, so π_ρ is weakly contained in $\hat{\pi}_u\iota_u$.

Since $\hat{\pi}_u\iota_u = \hat{\pi}|_{\hat{N}_c}\hat{\vartheta}^\natural$, we see that ρ is in $\mathrm{IR}(\hat{N}_c)$, where $\mathrm{IR}(\hat{N}_c)$ is considered as a subset of $\mathrm{IR}(\hat{N}_u)$ by the inclusion (3.14). Now we use Theorem 3.7.1 to identify $\mathrm{IR}(\hat{N}_c)$ with $\mathrm{IR}(\hat{M}_c, M_1)$, which we consider as a subspace of $\mathrm{IC}(M, M_1)$ by (3.14). We consider μ^{M_1} as a measure on $\mathrm{IC}(M, M_1)$ by defining the complement of $\mathrm{IR}(\hat{N}_c)$ in $\mathrm{IC}(M, M_1)$ to be negligible. Let $U_\sigma \in \mathrm{IC}(M, M_1)$ denote the corepresentation corresponding to $\sigma \in \mathrm{IR}(\hat{N}_c)$. So, $\pi_\sigma = \pi_{U_\sigma}^\natural$. We write $D_{U_\sigma}^{M_1}$ for $D_\sigma^{M_1}$ and set $D_U = 0$ for $U \in \mathrm{IC}(M, M_1)$ not in the support of μ^{M_1}. We denote $\mathcal{H}_U$ for the corepresentation Hilbert space of $U \in \mathrm{IC}(M, M_1)$, and we get $\mathcal{H}_{U_\sigma}^{M_1} = \mathcal{K}_\sigma^{M_1}$. Therefore, since the support of μ is contained in $\mathrm{IR}(\hat{N}_c)$, we see that $\mathcal{Q}_0^{M_1}$ is a map from $P_\beta P_\gamma \mathcal{H} \to \int_{\mathrm{IC}(M,M_1)}^\oplus \mathcal{H}_U^{M_1} \otimes \overline{\mathcal{H}_U^{M_1}} d\mu^{M_1}(U)$.

(1). It follows from Corollary 3.7.2, Remark 3.6.2 and the properties of $D_\sigma^{M_1}$ described in (1) of Theorem 1.6.2, that for $\omega \in \mathcal{I}_N$, the operator $(\tilde{\omega} \otimes \iota)(U)(D_U^{M_1})^{-1}$ is bounded and its closure is Hilbert-Schmidt for μ^{M_1}-almost every $U \in \mathrm{IC}(M, M_1)$.

(2) and (3). We make two observations. First note that by Proposition 3.4.6, for $\omega \in \mathcal{I}_N$, $\hat{\Lambda}_u\iota_u(\lambda_u^\natural(\omega)) = \xi(\tilde{\omega})$. Secondly, we have proved that for every ρ in the support of μ^{M_1}, there is a $U \in \mathrm{IC}(M, M_1)$, such that $\pi_\rho = \pi_U^\natural$. Then (2) of Theorem 1.6.2 yields (2) of the present theorem. The second observation also yields (3). Note that by Proposition 3.6.5, we see that $(\mathcal{H}_U)_{U\in\mathrm{IC}(M,M_1)}$ is a measurable field of Hilbert spaces of which $(\mathcal{H}_U^{M_1})_{U\in\mathrm{IC}(M,M_1)}$ forms a measurable field of subspaces. Here we use that $\hat{M}_u$ is separable.

To prove (4), we make the following observations. First of all, by Remark 3.7.3, we see that $\hat{\pi}_u\iota_u = \pi_{W_\mathcal{E}}^\natural$, where $W_\mathcal{E}$ denotes the restriction of the multiplicative unitary W to the Hilbert space $\mathcal{E}$. Secondly,

$$\int_{\mathrm{IC}(M,M_1)}^{\oplus} \pi_U^{\natural} \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu^{M_1}(U) = \pi^{\natural}_{\int_{\mathrm{IC}(M,M_1)}^{\oplus} U \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu(U)},$$

where we use Proposition 3.6.5 to infer that the direct integral on the right hand side exists. Hence, by Remark 3.6.7 we see that $W_{\mathcal{E}}$ and $\int_{\mathrm{IC}(M,M_1)}^{\oplus} U \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu^{M_1}(U)$ are equivalent. Moreover, we define $\mathcal{Q}^{M_1}$ to be the intertwiner as constructed in the proof of Proposition 3.6.6. Then, $\mathcal{Q}^{M_1}$ satisfies (4).

We now prove (5). The proof of Proposition 3.1.9 shows that $\hat{\sigma}_t(P_\gamma) = P_\gamma$. Write $D = \int_{\mathrm{IC}(M)}^{\oplus} D_U d\mu(U)$. Since D^{-2} is the Radon-Nikodym derivative of $\hat{\varphi}$ with respect to a trace on $\hat{M}$, see the proof of Theorem 1.6.2, we have $D\eta\hat{M}$ and $\hat{\sigma}_t(x) = D^{-2it}xD^{2it}$. Thus, P_γ commutes with D^{it} for all $t \in \mathbb{R}$. Since $P_\gamma \in \hat{M} \simeq \int_{\mathrm{IC}(M)}^{\oplus} B(\mathcal{H}_U) d\mu(U)$, we have a direct integral decomposition $P_\gamma = \int_{\mathrm{IC}(M)}^{\oplus} (P_\gamma)_U d\mu(U)$ and $(P_\gamma)_U$ commutes with D_U^{it} for μ-almost all $U \in \mathrm{IC}(M)$.

P_γ is the projection of $\mathcal{H}$ onto $\mathcal{H}^{M_1}$, see Proposition 3.5.2. A vector $v = \int_{\mathrm{IC}(M)}^{\oplus} v_U d\mu(U)$ is M_1-invariant for W if and only if v_U is M_1-invariant for μ-almost all $U \in \mathrm{IC}(M)$, as follows directly from Definition 3.5.1. Hence $(P_\gamma)_U$ is the projection of $\mathcal{H}_U$ onto $\mathcal{H}_U^{M_1}$ for μ-almost all $U \in \mathrm{IC}(M)$. We record three conclusions:

(P1) $\mathcal{H}_U^{M_1}$ is an invariant subspace of D_U for μ-almost all $U \in \mathrm{IC}(M)$;

(P2) $\mathrm{IC}(M, M_1)$ is a μ-measurable subset of $\mathrm{IC}(M)$ by Proposition A.1.3;

(P3) The image of $P_\gamma\mathcal{H}$ under $\mathcal{Q}$ equals $\int_{\mathrm{IC}(M,M_1)}^{\oplus} \mathcal{H}_U^{M_1} \otimes \overline{\mathcal{H}_U} d\mu(U)$.

For the choice of μ^{M_1}, $D_U^{M_1}$ and $\mathcal{Q}^{M_1}$ made in (5), properties (1) and (2) follow from the properties of μ, D_U and $\mathcal{Q}$ as described in Theorem 1.6.1 using Proposition 3.4.2.

By Theorem 1.6.1, $\mathcal{Q}$ intertwines W with $\int_{\mathrm{IC}(M)}^{\oplus} U \otimes 1_{\overline{\mathcal{H}_U}} d\mu(U)$. Using (P3) together with (3) of Theorem 1.6.2 and the fact that $P_\beta = \hat{J}P_\gamma\hat{J}$, we see that $\mathcal{Q}$ restricts to a unitary map from $P_\beta P_\gamma \mathcal{H}$ to $\int_{\mathrm{IC}(M,M_1)}^{\oplus} \mathcal{H}_U^{M_1} \otimes \overline{\mathcal{H}_U^{M_1}} d\mu(U)$. Hence $\mathcal{Q}$ restricts to a unitary map from $\mathcal{E}$ to $\int_{\mathrm{IC}(M,M_1)}^{\oplus} \mathcal{H}_U \otimes \overline{\mathcal{H}_U^{M_1}} d\mu(U)$, which then intertwines the restriction of W to $\mathcal{E}$ with $\int_{\mathrm{IC}(M,M_1)}^{\oplus} U \otimes 1_{\overline{\mathcal{H}_U^{M_1}}} d\mu(U)$. This proves (4), from which (3) follows by the construction in Remark 3.6.2 and Remark 3.7.3. $\square$

Remark 3.8.2. If $\Delta^{\natural}$ is cocommutative, then $\hat{N}$ is abelian. By Proposition 3.5.4 we see that for all $U \in \mathrm{IC}(M)$, $\dim(\mathcal{H}_U^{M_1}) \leq 1$. Hence the operators $D_U^{M_1}$ are scalars. We may assume that $D_U^{M_1} = 1$ by replacing the measure μ^{M_1} if necessary. In particular, we see that for a classical Gelfand pair (G, K), the map $\mathcal{Q}_0^{M_1}$ is the spherical Fourier transform.

Remark 3.8.3. The support of μ^{M_1} is given by $\mathrm{IR}(\hat{N}_c)$. Here, $\mathrm{IR}(\hat{N}_c)$ is a subspace of IC(M, M_1) as in (3.14). The prove can be done in exactly the same manner as [19, Theorem 3.4.8], see also [22, Proposition 8.6.8]. Note that in the course of the proof of Theorem 3.8.1 we have already proved that the support of μ^{M_1} is contained in $\mathrm{IR}(\hat{N}_c)$.

Remark 3.8.4. As pointed out in Remark 1.6.4, the Theorem 3.8.1 also holds if one assumes that $\hat{N}_c$ and $\hat{M}_c$ are separable, instead of $\hat{N}_u$. Moreover, note that if $\hat{M}_c$ is separable, then so is $\hat{N}_c \subseteq \hat{M}_c$. If $\hat{M}$ is a type I von Neumann algebra, then so is $\hat{N} = P_\gamma \hat{M} P_\gamma$. In particular, if $\hat{M}$ is type I and $\hat{M}_c$ is separable, then then the result of Theorem 3.8.1 holds for any closed quantum subgroup of (M, Δ).

Chapter 4

Gelfand pair properties of $(SU_q(1,1)_{\text{ext}}, \mathbb{T})$

Let (M, Δ) be the quantum group analogue of the normaliser of $SU(1,1)$ in $SL(2,\mathbb{C})$ as introduced in Section 1.8. In this chapter we identify the circle as a closed quantum subgroup of (M, Δ) and make all the objects of Chaper 3 precise. We show that for this pair the map $\Delta^\natural$ defined in (3.3) is not cocommutative. Moreover, the von Neumann subalgebra N of M consisting of bi-invariant elements is not commutative. However, we show how the von Neumann algebras N and $\hat{N}$ as defined in Chapter 3 can be equipped with a $\mathbb{Z}_2$-grading. The grading allows us to derive similar results as for (quantum) Gelfand pairs. In particular, we make the Fourier transform explicit and show that it preserves the $\mathbb{Z}_2$-grading. Moreover, we derive product formulae for little q-Jacobi functions appearing as matrix coefficients of corepresentations which admit invariant vectors.

The contents of this chapter is published in the forthcoming paper [8].

Remark 4.0.1. In Section 2.4 it is proved that $SU_q(1,1)_{\text{ext}}$ satisfies the hypotheses of the Plancherel theorem, Theorem 1.6.1. By Remark 3.8.4 we can also apply Theorem 3.8.1.

Notation 4.0.2. In this chapter we adopt all the notational conventions made in [35]. These are recalled in Section 1.8. In particular, in this chapter we write $\mathcal{K}$ instead of $\mathcal{H}$ to denote the GNS-space of (M, Δ). For $z \in \mathbb{C}$, we denote $\mu(z) = (z + z^{-1})/2$. For a set X and $x \in X$, we will write δ_x for the function on X that equals 1 in x and 0 elsewhere. It should always be clear from the context what the domain of this function is. Recall in particular that $I_q = -q^{\mathbb{N}} \cup q^{\mathbb{Z}}$, where $\mathbb{N}$ denotes the natural numbers excluding 0.

4.1 The diagonal subgroup

Let $\mathbb{T}$ denote the circle group and let $(L^\infty(\mathbb{T}), \Delta_{\mathbb{T}})$ be the usual locally compact quantum group associated with $\mathbb{T}$ with dual quantum group $(L^\infty(\mathbb{Z}), \Delta_{\mathbb{Z}})$. We identify $(L^\infty(\mathbb{T}), \Delta_{\mathbb{T}})$ as a closed quantum subgroup of (M, Δ). Recall from (1.18) that the spectrum of the operator K equals $\{0\} \cup q^{\frac{1}{2}\mathbb{Z}}$.

Definition 4.1.1. We define a normal, injective $*$-homomorphism $\hat{\pi} : L^\infty(\mathbb{Z}) \to \hat{M}$ by setting $\hat{\pi}(\delta_k) = \delta_{q^{\frac{k}{2}}}(K)$.

Note that $\hat{\pi}$ preserves the comultiplication, since

$$(\hat{\pi} \otimes \hat{\pi})\Delta_{\mathbb{Z}}(\delta_k) = (\hat{\pi} \otimes \hat{\pi}) \left(\sum_{l \in \mathbb{Z}} \delta_l \otimes \delta_{k-l} \right)$$

$$= \sum_{l \in \mathbb{Z}} \delta_{q^{\frac{l}{2}}}(K) \otimes \delta_{q^{\frac{k-l}{2}}}(K) = \delta_{q^{\frac{k}{2}}}(K \otimes K) = \delta_{q^{\frac{k}{2}}}(\hat{\Delta}(K)).$$

Therefore, $\hat{\pi}$ identifies $(\mathbb{T}, \Delta_{\mathbb{T}})$ as a closed quantum subgroup of (M, Δ), see Definition 1.3.3. Furthermore, $\hat{\pi}$ induces a morphism π between the universal quantum groups (M_u, Δ_u) and $(C(\mathbb{T}), \Delta_{\mathbb{T}})$, where here with slight abuse of notation $\Delta_{\mathbb{T}}$ is restricted to a map $C(\mathbb{T}) \to C(\mathbb{T} \times \mathbb{T})$.

4.2 Spherical corepresentations

We compute the actions γ and β of left and right translation and determine which of the corepresentations found in [35] (see Section 1.8) admit a $L^\infty(\mathbb{T})$-invariant vector.

Proposition 4.2.1. *For all $p \in q^{\mathbb{Z}}, x \in (\mu(-q), \mu(q)) \cup \sigma_d(\Omega_p)$,*

$$\beta(\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) = (\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) \otimes \zeta^{2m}, \tag{4.1}$$

$$\gamma(\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) = (\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) \otimes \zeta^{2m'}. \tag{4.2}$$

Here, ζ is the identity function on the complex unit circle $\mathbb{T}$.

Proof. Note that $\hat{\pi}$ has a direct integral decomposition $\hat{\pi} = \int^\oplus \hat{\pi}_{p,x} d(p,x)$. Here $\hat{\pi}_{p,x} : L^\infty(\mathbb{Z}) \to B(\mathcal{L}_{p,x})$ is determined by $\hat{\pi}_{p,x}(\delta_k) = \delta_{q^{\frac{k}{2}}}(K)$, where the action of K on the representation space is given by (1.28) and similarly for the discrete series corepresentations.

By the definition of β, see (3.1), and the decomposition (1.22), we see that

$$\int^\oplus (\beta \otimes \iota)(W_{p,x}) d(p,x) = \int^\oplus (W_{p,x})_{13} (\iota \otimes \hat{\pi}_{p,x})(W_{\mathbb{T}})_{23} d(p,x).$$

By (1.28) and the definition of $\hat{\pi}$, for almost all pairs (p,x) in the decomposition (1.22),

$$(\beta \otimes \iota)(W_{p,x}) = (W_{p,x})_{13}(\iota \otimes \hat{\pi}_{p,x})(W_{\mathbb{T}})_{23}. \tag{4.3}$$

This proves (4.1) for the discrete series corepresentations. It follows from (1.24) and the fact that the function C given there is analytic on a neighbourhood of $(\mu(-q),\mu(q))$, see [35, Section 10.3], that for every $p \in q^{\mathbb{Z}}$,

$$(\mu(-q),\mu(q)) \to M \otimes L^{\infty}(\mathbb{T}) : x \mapsto \beta\left((\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x})\right), \tag{4.4}$$

extends to an analytic function on a neighbourhood of $(\mu(-q),\mu(q))$. Here, we use the fact that β is normal and the fact that a function is σ-weak analytic if and only if it is analytic with respect to the norm, see Proposition A.2.1. Similarly, it follows that for $p \in q^{\mathbb{Z}}$,

$$(\mu(-q),\mu(q)) \to M \otimes L^{\infty}(\mathbb{T}) : x \mapsto (\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) \otimes \zeta^{2m}, \tag{4.5}$$

extends to an analytic function on a neighbourhood of $(\mu(-q),\mu(q))$. Note that for all $p \in q^{\mathbb{Z}}, x \in (\mu(-q),\mu(q))$,

$$\begin{aligned}
&(\iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})(W_{p,x}) \otimes \zeta^{2m} \\
=&(\iota \otimes \iota \otimes \omega_{f_m^{\varepsilon,\eta}(p,x), f_{m'}^{\varepsilon',\eta'}(p,x)})\left((W_{p,x})_{13}(\iota \otimes \hat{\pi}_{p,x})(W_{\mathbb{T}})_{23}\right).
\end{aligned}$$

Now, (4.3) yields that (4.4) and (4.5) agree on a dense subset of $[-1,1]$. Since (4.4) and (4.5) have analytic extensions, they are equal for all $x \in (\mu(-q),\mu(q))$. The proof of (4.2) is similar. $\square$

Recall that we defined $L^{\infty}(\mathbb{T})$-invariant vectors in Definition 3.5.1.

Corollary 4.2.2. *Let $p \in q^{\mathbb{Z}}, x \in (\mu(-q),\mu(q)) \cup \sigma_d(\Omega_p)$.*

1. *If $p \in q^{2\mathbb{Z}}$, then the space of $L^{\infty}(\mathbb{T})$-invariant vectors of $W_{p,x}$ is spanned by $f_0^{\varepsilon,\eta}(p,x)$, with $\varepsilon,\eta \in \{+,-\}$.*

2. *If $p \in q^{2\mathbb{Z}}, x \in \sigma_d(\Omega_p)$. The space of $L^{\infty}(\mathbb{T})$-invariant vectors of $W_{p,x}$ is one dimensional.*

3. *If $p \in q^{1+2\mathbb{Z}}$, then $W_{p,x}$ has no $L^{\infty}(\mathbb{T})$-invariant vectors.*

4. *$P_\gamma = \delta_1(K) = \hat{\pi}(\delta_0)$ and hence the von Neumann algebras N and $\hat{N}$ constructed in Sections 3.1 and 3.3 are given by*

$$\begin{aligned}
N &= \overline{\left\{(\iota \otimes \omega_{\delta_1(K)v,\delta_1(K)w})(W) \mid v,w \in \mathcal{K}\right\}}^{\sigma-\text{strong}-*}, \\
\hat{N} &= \delta_1(K)\hat{M}\delta_1(K).
\end{aligned}$$

Proof. From the considerations in Chapter 3, (1), (3) and (4) follow. For (2) consider $\lambda \in -q^{2\mathbb{Z}+1} \cup q^{2\mathbb{Z}+1}$ be such that $x = \mu(\lambda)$ and $|\lambda| \geq 1$. Put $j', l' \in \mathbb{Z}$ by setting $|\lambda| = q^{1-2j'} = q^{1+2l'}$. In particular, $l' < j'$. For case (i) of Proposition 1.8.9, $f_0^{-,+}(p,x)$ is zero if and only if $0 > l'$ if and only if $|\lambda| \geq 1$, which is true by assumption. Similarly, $f_0^{+,-}(p,x) = 0$. Hence the only $L^\infty(\mathbb{T})$-invariant vector is $f_0^{+,+}(p,x)$. Cases (ii) and (iii) of Proposition 1.8.9 follow in exactly the same manner. □

Remark 4.2.3. Note that the space of $L^\infty(\mathbb{T})$-invariant vectors for the irreducible components of $W_{p,x}$ is not necessarily of dimension ≤ 1. For example $W_{p,x}, p \in q^{2\mathbb{Z}}, x \in [-1,1]\backslash\{0\}$ splits as a sum of 2 irreducible corepresentations of which the $L^\infty(\mathbb{T})$-invariant vectors form a 2-dimensional vector space, see [35, Section 10.2] or Section 1.8. This implies that $\Delta^\natural$ is not cocommutative and we can not use (3.6) directly to obtain product formulae. However, we define gradings on the spaces N and $\hat{N}$ which still allows us to find such formulae.

In Corollary 4.2.2 we have determined the corepresentations appearing in the decomposition that admit a $L^\infty(\mathbb{T})$-invariant vector. These corepresentations are not mutually inequivalent. Here, we give a list of the equivalences. We only consider the spherical corepresentations and consider the principal, discrete as well as the complementary series. Only the principal and discrete series are important to determine the spaces N, $\hat{N}$ and the spherical Fourier transform, see Theorem 3.8.1. Nevertheless, the complementary series is still important, since the product formulae we derive later still hold for the spherical matrix elements of the complementary series.

In Section 1.8 that we have defined the basis vectors $f_m^{\varepsilon,\eta}(p,x)$. For $p \in q^{2\mathbb{Z}}, x \in (\mu(-q), \mu(q))$, set

$$\begin{aligned}
g_m^{1,+}(p,x) &= \frac{1}{2}\sqrt{2}(f_m^{+,+}(p,x) + i^{\chi(p)} f_m^{-,-}(p,x)),\\
g_m^{1,-}(p,x) &= \frac{1}{2}\sqrt{2}(f_m^{+,-}(p,x) - i^{\chi(p)} f_m^{-,+}(p,x)),\\
g_m^{2,+}(p,x) &= \frac{1}{2}\sqrt{2}(f_m^{+,+}(p,x) - i^{\chi(p)} f_m^{-,-}(p,x)),\\
g_m^{2,-}(p,x) &= \frac{1}{2}\sqrt{2}(f_m^{+,-}(p,x) + i^{\chi(p)} f_m^{-,+}(p,x)).
\end{aligned} \tag{4.6}$$

For every $j \in \{1,2\}, p \in q^{2\mathbb{Z}}, x \in (\mu(-q), \mu(q))$, the vectors $g_m^{j,\sigma}(p,x), \sigma \in \{+,-\}, m \in \mathbb{Z}$ form an orthonormal basis for $\mathcal{L}_{p,x}^j$, the corepresentation space of one of the summands of $W_{p,x}$, see [35, Eqn. (95)] or Section 1.8. For $p \in q^{2\mathbb{Z}}, x \in \sigma_p(\Omega_d)$, set

$$\begin{aligned}
g_m^{1,+}(p,x) &= f_m^{+,+}(p,x) + i^{\chi(p)} f_m^{-,-}(p,x),\\
g_m^{1,-}(p,x) &= f_m^{+,-}(p,x) - i^{\chi(p)} f_m^{-,+}(p,x),\\
g_m^{2,+}(p,x) &= f_m^{+,+}(p,x) - i^{\chi(p)} f_m^{-,-}(p,x),\\
g_m^{2,-}(p,x) &= f_m^{+,-}(p,x) + i^{\chi(p)} f_m^{-,+}(p,x).
\end{aligned} \tag{4.7}$$

Recall the convention that $f_m^{\varepsilon,\eta}(p,x) = 0$ in case $f_m^{\varepsilon,\eta}(p,x)$ is not in the basis given in Proposition 1.8.9. Hence, for $x \in \sigma_d(\Omega_p)$, we see from Proposition 1.8.9 that the vectors defined in (4.7) are dependent and any of them is equal to a vector of the form $f_m^{\varepsilon,\eta}(p,x)$ for some $\varepsilon, \eta \in \{+,-\}$ up to a phase factor.

Now we determine which of the discrete, principal and complementary series corepresentations are equivalent. By considering the action of the Casimir operator as in the proof of Proposition 2.3.2 it is clear that any two corepresentations that fall within a different series are inequivalent. We restrict ourselves to the spherical corepresentations.

Proposition 4.2.4. *1. Let $x, x' \in (\mu(-q), \mu(q))\backslash\{0\}, p, p' \in q^{2\mathbb{Z}}, j, j' \in \{1,2\}$. $W_{p,x}^{j} \simeq W_{p',x'}^{j'}$ if and only if either $x = \pm x', j = j', p/p' \in q^{4\mathbb{Z}}$ or $x = \pm x', j \neq j', p/p' \in q^{2+4\mathbb{Z}}$.*

2. Let $p, p' \in q^{2\mathbb{Z}}, j, j', k, k' \in \{1,2\}$. $W_{p,0}^{j,k} \simeq W_{p',0}^{j',k'}$ if and only if either $j = j', k = k', p/p' \in q^{4\mathbb{Z}}$ or $j \neq j', k = k', p/p' \in q^{2+4\mathbb{Z}}$.

3. Let $x, x' \in \mu(-q^{\mathbb{Z}} \cup q^{\mathbb{Z}}), p, p' \in q^{2\mathbb{Z}}$. $W_{p,x} \simeq W_{p',x'}$ if and only if $|x| = |x'|$.

Proof. The proposition follows from a careful comparison of the action of the generators, see (1.28) for the principal series, [35, Proposition 5.2 and Lemma 10.1] for the discrete series and [35, Section 10.3] for the complementary series. We prove (1). By considering the Casimir operator (1.19) one sees that if an irreducible component of $W_{p,x}$ is equivalent to an irreducible component of $W_{p',x'}$, then $|x| = |x'|$.

In case $x = x'$, an intertwiner must send $g_m^{j,\pm}(p,x)$ to a non-zero scalar multiple of $g_m^{j',\pm}(p',x)$ as follows by considering the actions of K and E. Writing out the actions of U_0^{+-} and U_0^{-+} one sees that there exists such an intertwiner only in the following two cases:

(i) $p/p' \in q^{4\mathbb{Z}}, j = j'$, for which it sends $g_m^{j,\pm}(p,x)$ to $g_m^{j,\pm}(p',x)$;

(ii) $p/p' \in q^{2+4\mathbb{Z}}, j \neq j'$, in which case it sends $g_m^{j,\pm}(p,x)$ to $\pm g_m^{j',\pm}(p',x)$.

Similarly, in case $x = -x'$, an intertwiner must send $g_m^{j,\pm}(p,x)$ to $g_m^{j',\mp}(p',-x)$ as follows from the actions of K and E. The actions of U_0^{+-} and U_0^{-+} show that this is only possible if

(i) $p/p' \in q^{4\mathbb{Z}}, j = j'$ for which it sends $g_m^{j,\pm}(p,x)$ to $\pm g_m^{j,\mp}(p',-x)$;

(ii) $p/p' \in q^{2+4\mathbb{Z}}, j \neq j'$, in which case it sends $g_m^{j,\pm}(p,x)$ to $g_m^{j',\pm}(p',-x)$.

This proves (1), the other cases follows similarly. □

Summarizing Corollary 4.2.2 and Proposition 4.2.4, we find that IC(M, M_1), the space of equivalence classes of irreducible spherical corepresentations is partly given by.

$$\begin{array}{lll} \mathrm{IC}(M, M_1) & \supseteq & \\ & & \\ \multicolumn{2}{l}{(0,1,1) \cup (0,1,2) \cup (0,2,1) \cup (0,2,2) \cup ((0,1] \times \{1,2\})} & \text{(principal)} \\ \cup \quad \mu(q^{2\mathbb{N}+1}) & & \text{(discrete)} \\ \cup \quad (1, \mu(q)) \times \{1,2\} & & \text{(complementary)} \end{array} \tag{4.8}$$

Here, we identify the points $(0,1,1), (0,1,2), (0,2,1), (0,2,2)$ with the respective irreducible corepresentations $W_{1,0}^{1,1}, W_{1,0}^{1,2}, W_{1,0}^{2,1}, W_{1,0}^{2,2}$, see [35, Proposition 10.14 (ii)] or Section 1.8. We let a point $(x, j) \in (0,1] \times \{1,2\}$ correspond to $W_{1,x}^{j}$, see [35, Proposition 10.12] or Section 1.8. The points $x \in \mu(q^{2\mathbb{N}+1})$ corresponds to $W_{1,x}$, see Proposition 1.8.9. We emphasize that it is not known if the corepresentations described in [35] are all the corepresentations, therefore we do not know if this description completely describes $\mathrm{IC}(M, M_1)$. For the von Neumann algebras N and $\hat{N}$ as well as the spherical Fourier transform, only the principal and discrete (spherical) series matter. These are fully identified within (4.8). For completeness, we illustrate the right hand side of (4.8) by means of Figure 4.1.

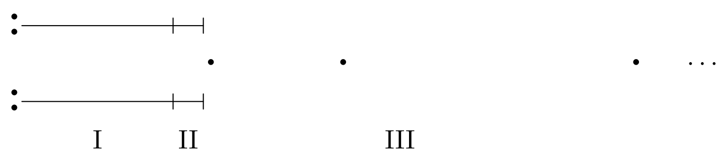

Figure 4.1: Known part of $\mathrm{IC}(M, M_1)$ for $SU_q(1,1)_{\text{ext}}$ with the diagonal subgroup, c.f. (4.8). Part I: Principal spherical series. Part II: Complementary spherical series. Part III: Discrete spherical series.

4.3 The von Neumann algebra N

The next step is to make N and $\hat{N}$ more explicit and meanwhile define a grading on these spaces. Therefore, we first find an alternative formula for the mapping $T_\beta T_\gamma$ which is convenient for computations. This is also going to play a role when we derive product formulae. Recall that the spectrum of K is given by $q^{\frac{1}{2}\mathbb{Z}}$. For $k, l \in \mathbb{Z}$, set

$$M_{k,l} = \overline{\left\{ (\iota \otimes \omega_{\delta_{q^{\frac{1}{2}(k-l)}}(K)v, \delta_{q^{\frac{1}{2}(k+l)}}(K)w})(W) \mid v, w \in \mathcal{K} \right\}}^{\sigma-\text{strong}-*}.$$

By (1.28) and (1.24), the spaces $M_{k,l}$ have mutually trivial intersection. Moreover, M equals the σ-strong-$*$ closure of the direct sum of vector spaces $\oplus_{k,l\in\mathbb{Z}} M_{k,l}$. Note that $N = M_{0,0}$.

Definition 4.3.1. For $m \in \mathbb{Z}, p, t \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$, let $P_{m,p,t}$ be the orthogonal rank one projection of $\mathcal{K}$ onto the space spanned by the vector $f_{m,p,t}$. Here we define $f_{m,p,t}$ to be the zero vector if either $p \notin I_q$ or $t \notin I_q$. Define mappings T^+ and T^- defined by

$$T^{\pm} : M \to M : x \mapsto \sum_{m\in\mathbb{Z}, p,t\in I_q} P_{m,\pm p,t} x P_{m,p,t},$$

where the sum converges in the strong topology.

Proposition 4.3.2. $T^+ + T^- = T_\beta T_\gamma$.

Proof. For a normal functional $\omega = \sum_{i\in I} \omega_{\xi_i,\eta_i} \in M_*$, with $\sum_{i\in I} \|\xi_i\|^2 < \infty$ and $\sum_{i\in I} \|\eta_i\|^2 < \infty$, the linear map

$$M \to \mathbb{C} : x \mapsto \omega T^{\pm}(x) = \sum_{i\in I} \sum_{m\in\mathbb{Z}, p,t\in I_q} \langle x P_{m,p,t} \xi_i, P_{m,\pm p,t} \eta_i \rangle,$$

is normal. Hence, the maps $T^{\pm}$ are normal. Moreover, for $x \in M_{k,l}$, using (1.24),

$$T^+(x) + T^-(x) = \begin{cases} x & \text{if } k = l = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

For $x \in M_{k,l}$ we see by using (4.1) and (4.2) that $T_\beta T_\gamma(x)$ also equals the right hand side of (4.9). Since also $T_\beta T_\gamma$ is normal, this proves that $T^+ + T^- = T_\beta T_\gamma$. $\square$

Let $u_0 \in B(L^2(I_q))$ be the partial isometry determined by $u_0 : \delta_p \mapsto \delta_{-p}$, and recall that u was defined as $u = 1 \otimes u_0 \in B(L^2(\mathbb{Z}) \otimes L^2(I_q))$. By [51, Lemma 2.4] $M \simeq L^\infty(\mathbb{T}) \otimes B(L^2(I_q))$. Proposition 4.3.2 yields,

$$N = T_\beta T_\gamma(M) = (T^+ + T^-)(M),$$

which is isomorphic to the von Neumann subalgebra of M generated by $1 \otimes L^\infty(I_q)$ and u. Therefore we introduce the following identifications.

Definition 4.3.3. We identify N with the von Neumann algebra acting on $L^2(I_q)$ being generated by $L^\infty(I_q)$ and u_0. We split N as a direct sum of vector spaces

$$N = N_+ \oplus N_-, \text{ where } N_+ = L^\infty(I_q), \text{ and } N_- = L^\infty(I_q)u_0 = L^\infty(I_q \cap (-1,1))u_0.$$

This turns N into a $\mathbb{Z}_2$-graded algebra.

We find that $\varphi^\natural$, i.e. the restriction of the Haar weight φ to N, equals the measure given by:

$$\varphi^\natural(f) = \sum_{p_0 \in I_q} f(p_0) p_0^2, \qquad f \in N^+ = L^\infty(I_q)^+.$$

For $f \in \mathfrak{m}^+_{\varphi^\natural}$ we find that $u_0 f \in \mathfrak{m}_{\varphi^\natural}$ and it follows that $\varphi^\natural(u_0 f) = 0$, so that $\Lambda(\mathfrak{n}_\varphi \cap N_+)$ and $\Lambda(\mathfrak{n}_\varphi \cap N_-)$ are orthogonal spaces. The discussion so far allows us to make the following identifications.

Definition 4.3.4. Identify the closure of $\Lambda(\mathfrak{n}_\varphi \cap N_+)$ with $L^2(I_q)$. We write $L^2(N_+)$ for $L^2(I_q)$ to indicate explicitly that $L^2(I_q)$ is considered as part of the GNS-space of $\varphi^\natural$. Similarly, we identify the closure of $\Lambda(\mathfrak{n}_\varphi \cap N_-)$ with $L^2(I_q \cap (-1,1))$, by identifying $f u_0 \in N_-$, where $f \in L^\infty(I_q \cap (-1,1)) \cap L^2(I_q \cap (-1,1))$. We write short hand $L^2(N_-)$ for $L^2(I_q \cap (-1,1))$. We write $L^2(N) = L^2(N_+) \oplus L^2(N_-)$. We emphasize that here the spaces $L^2(I_q)$, respectively $L^2(I_q \cap (-1,1))$, should be understood with respect to the integral given by the weighted sum $\sum_{p_0 \in I_q} (\ \cdot\ ) p_0^2$, respectively $\sum_{p_0 \in I_q \cap (-1,1)} (\ \cdot\ ) p_0^2$.

4.4 The von Neumann algebra $\hat{N}$

We now turn our attention to the von Neumann algebra $\hat{N}$ as defined in Section 3.3. Considered as a subalgebra of $\hat{M}$, $\hat{N}$ inherits the $\mathbb{Z}_2$-grading defined in Definition 1.8.12.

Definition 4.4.1. We set

$$\hat{N}_+ = \hat{N} \cap \hat{M}_+, \qquad \hat{N}_- = \hat{N} \cap \hat{M}_-,$$

see Definition 1.8.12 for $\hat{M}_+$ and $\hat{M}_-$. This turns $\hat{N}$ into a $\mathbb{Z}_2$-graded algebra.

Proposition 4.4.2. *We have an isomorphism of von Neumann algebras:*

$$\hat{N} \simeq \int_{[0,1]}^{\oplus} M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) dx \oplus \bigoplus_{x \in \mu(q^{2\mathbb{N}+1})} \mathbb{C}. \tag{4.10}$$

The isomorphism is determined by the map

$$(\tilde{\omega} \otimes \iota)(W) \mapsto \int_{[0,1]}^{\oplus} (\tilde{\omega} \otimes \iota)(W^1_{1,x}) \oplus (\tilde{\omega} \otimes \iota)(W^2_{1,x}) dx \oplus \bigoplus_{x \in \mu(q^{2\mathbb{N}+1})} (\tilde{\omega} \otimes \iota)(W_{1,x}). \tag{4.11}$$

Under this isomorphism $\hat{N}_+$ corresponds to the direct integrals over matrices whose entries vanish off the diagonal. $\hat{N}_-$ corresponds to the direct integrals over matrices with values vanishing on the diagonal.

Proof. It follows from Corollary 4.2.2 that the map defined in (4.11) indeed maps into the proper matrix algebras. Let $P \in \hat{M} \cap \hat{M}'$ be the projection as in the proof of Proposition 2.3.3. By considering the Casimir operator, one sees that P is the spectral projection on the interval $[-1, 1]$ of the Casimir operator. In Proposition 2.3.3, it is shown that

$$P\hat{M}P \simeq \int_{x\in[0,1]}^{\oplus} \hat{M}_x dx,$$

where $\hat{M}_x$ is generated by $\{(\omega \otimes \iota)(W_x) \mid \omega \in M_*\}$ and $W_x = (\oplus_{p\in q^{\mathbb{Z}}} W_{p,x}) \oplus (\oplus_{p\in q^{\mathbb{Z}}} W_{p,-x})$. By similar techniques as in Proposition 2.3.3, one can show that

$$\hat{M} \simeq \int_{x\in[0,1]}^{\oplus} \hat{M}_x dx \oplus \bigoplus_{x\in\sigma_d(\Omega)\cap(1,\infty)} \hat{M}_x, \tag{4.12}$$

where $\hat{M}_x, x \in \sigma_d(\Omega) \cap (1, \infty)$ is generated by $\{(\omega \otimes \iota)(W_x) \mid \omega \in M_*\}$ and

$$W_x = (\bigoplus_{p\in q^{\mathbb{Z}},\text{s.t.}\, x\in\sigma_d(\Omega_p)} W_{p,x}) \oplus (\bigoplus_{p\in q^{\mathbb{Z}},\text{s.t.}\, -x\in\sigma_d(\Omega_p)} W_{p,-x}).$$

Since $P_\gamma \in \hat{M}$, it has a direct integral decomposition as in (4.12), i.e.

$$P_\gamma \hat{M} P_\gamma \simeq \int_{x\in[0,1]}^{\oplus} (P_\gamma)_x \hat{M}_x (P_\gamma)_x dx \oplus \bigoplus_{x\in\sigma_d(\Omega)\cap(1,\infty)} (P_\gamma)_x \hat{M}_x (P_\gamma)_x.$$

Similarly, K decomposes with respect to the direct integral decomposition (4.12) as a direct integral over a field $(K_x)_{x\in[0,1]\cup(\sigma_d(\Omega)\cap(1,\infty))}$ and by Proposition 4.2.1, we see that $(T_\beta T_\gamma \otimes \iota)(W_x) = (1 \otimes \delta_1(K_x))W_x(1 \otimes \delta_1(K_x)) = (1 \otimes (P_\gamma)_x)W_x(1 \otimes (P_\gamma)_x)$. Hence, $(P_\gamma)_x \hat{M}_x (P_\gamma)_x$ is generated by $\{(\tilde{\omega} \otimes \iota)(W_x) \mid \omega \in M_*\}$. Due to Corollary 4.2.2 and Proposition 4.2.4, the latter von Neumann algebra is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ in case $x \in (0, 1]$ and to $\mathbb{C}$ in case $x \in \sigma_d(\Omega) \cap (1, \infty)$. This proves (4.10). That (4.11) gives the isomorphism follows directly from this proof. The claim on the gradings follows from Proposition 1.8.13. □

Remark 4.4.3. The von Neumann algebra $\hat{M}_0$ in the previous proof is isomorphic to 4 copies of $\mathbb{C}$ as follows from [35, Proposition 10.13]. Since this does not matter for the integral decomposition (4.10), and in order to avoid some redundant extra notation we have not treated the point $x = 0$ separately.

We claim that $\hat{\varphi}^\natural$, i.e. the restriction of the dual left Haar weight to $\hat{N}$, is a trace. More specific, it follows from Corollary 2.4.8 that there exists a measurable (scalar valued) function $e(x), x \in [0, 1] \cup \mu(q^{2\mathbb{N}+1})$ such that:

$$\hat{\varphi}^\natural = \int_{[0,1]}^{\oplus} \left(\mathrm{Tr}_{M_2(\mathbb{C})} \oplus \mathrm{Tr}_{M_2(\mathbb{C})}\right) e(x) dx \oplus \bigoplus_{x\in\mu(q^{2\mathbb{N}+1})} e(x)\mathrm{Tr}_{\mathbb{C}}. \tag{4.13}$$

In fact, for $x \in [0,1]$, $e(x)dx = d_0^{-2} d\mu(x)$, where μ is the restriction of the Askey-Wilson measure (2.25) to the interval $[0,1]$ and d_p was computed in Theorem 2.4.6.

We identify the GNS-space of $\varphi^\natural$ using the isomorphism (4.10). That is, the GNS-space is given by

$$L^2(\hat{N}) = \int_{[0,1]}^{\oplus} M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) dx \oplus \bigoplus_{x \in \mu(q^{2\mathbb{N}+1})} \mathbb{C}, \tag{4.14}$$

where the direct integral and direct sums are taken as Hilbert spaces (as opposed to the direct integrals and sums of von Neumann algebras given in (4.10)), and the inner product comes from the traces $e(x)(\mathrm{Tr}_{M_2(\mathbb{C})} \oplus \mathrm{Tr}_{M_2(\mathbb{C})})$ for the integral part and $e(x)\mathrm{Tr}_{\mathbb{C}}$ for the direct summands.

It follows from (4.10) and (4.13) that $\hat{\Lambda}(\mathfrak{n}_{\hat{\varphi}} \cap \hat{N}_+)$ and $\hat{\Lambda}(\mathfrak{n}_{\hat{\varphi}} \cap \hat{N}_-)$ are orthogonal spaces, we denote their closures interpreted within $L^2(\hat{N})$ by $L^2(\hat{N}_+)$ and $L^2(\hat{N}_-)$. They consist of direct integrals of diagonal matrices and off-diagonal matrices, respectively.

4.5 The spherical Fourier transform

Here, we determine the spherical Fourier transform, i.e. we determine the map $\mathcal{Q}_0^{L^\infty(\mathbb{T})}$ from Theorem 3.8.1. In particular, we are interested in the kernels of the integral transformations that appear in this transform, since we need them later. Using the identifications of the GNS-spaces for $\varphi^\natural$ and $\hat{\varphi}^\natural$ with $L^2(N)$ and $L^2(\hat{N})$, we may consider $\mathcal{Q}_0^{L^\infty(\mathbb{T})}$ as a map from $L^2(N)$ to $L^2(\hat{N})$ and use the short hand notation $\mathcal{F}_2 : L^2(N) \to L^2(\hat{N})$ for this map.

If $f \in L^1(I_q) \cap L^\infty(I_q) \subseteq N_+$, then there is a functional $f \cdot \varphi^\natural \in N_*$ given by $(f \cdot \varphi^\natural)(x) = \varphi^\natural(xf), x \in N$. Then, $\xi(f \cdot \varphi^\natural) = \Lambda(f)$ by definition of ξ, see (1.5). Thus for such f, we find by Theorem 3.8.1 that,

$$\mathcal{F}_2 : \Lambda(f) \mapsto ((f \cdot \varphi^\natural)^\sim \otimes \iota) \left(\int_{[0,1]}^{\oplus} W_{1,x} dx \oplus \bigoplus_{x \in \mu(q^{2\mathbb{N}+1})} W_{1,x} \right). \tag{4.15}$$

Here, the right hand side indeed is an element of $L^2(\hat{N})$ by Corollary 4.2.2. Note that we choose the inner product on $L^2(\hat{N})$ te be the direct integral over the traces $e(x)(\mathrm{Tr}_{M_2(\mathbb{C})} \oplus \mathrm{Tr}_{M_2(\mathbb{C})})$ for the integral part and $e(x)\mathrm{Tr}_{\mathbb{C}}$ for the direct summands. Hence, the Duflo-Moore operators which are direct integrals of scalar multiples of the identity are contained in the inner product by means of the function e. Hence, they do not appear in (4.15). Similarly, for $f \in L^1(I_q) \cap L^\infty(I_q)$, so that $fu_0 \in N_-$,

$$\mathcal{F}_2 : \Lambda(fu_0) \mapsto ((fu_0 \cdot \varphi^\natural)^\sim \otimes \iota) \left(\int_{[0,1]}^{\oplus} W_{1,x} dx \oplus \bigoplus_{x \in \mu(q^{2\mathbb{N}+1})} W_{1,x} \right). \tag{4.16}$$

We see from (4.15) that the spherical Fourier transformation is in fact a combination of integral transformations with the spherical matrix elements of the corepresentations $W_{1,x}$ in its kernel. The next step is to make these kernels explicit.

For any $m_0 \in \mathbb{Z}, t_0 \in I_q, x \in [-1,1] \cup \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1}), j \in \{1,2\}, \sigma, \tau \in \{+,-\}$, put

$$K_j^{\sigma,\tau}(p_0; x) = \langle (\iota \otimes \omega_{g_0^{j,\sigma}(1,x), g_0^{j,\tau}(1,x)})(W_{1,x}) f_{m_0,p_0,t_0}, f_{m_0,\sigma\tau p_0,t_0} \rangle. \tag{4.17}$$

For $x \in [-1,1] \cup \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1})$ but $x \notin \sigma_d(\Omega_1)$, the corepresentation $W_{1,x}$ is defined by Remark 1.8.11. In turn this defines $K_j^{\sigma,\tau}(p_0; x)$ for any $x \in [-1,1] \cup \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1})$. This expression is independent of m_0 and t_0. Due to the proof of Proposition 4.2.4, we have the following symmetry

$$K_j^{\sigma,\tau}(p_0; x) = \sigma\tau K_j^{-\sigma,-\tau}(p_0; -x). \tag{4.18}$$

From (1.24), (4.15) and (4.16), we see that we get a graded Fourier transform $\mathcal{F}_{2,+} \oplus \mathcal{F}_{2,-} : L^2(N_+) \oplus L^2(N_-) \to L^2(\hat{N}_+) \oplus L^2(\hat{N}_-)$, which is defined by sending $f \oplus g u_0 \in (L^2(I_q) \cap L^1(I_q)) \oplus (L^2(I_q \cap (-1,1)) \cap L^1(I_q \cap (-1,1))) u_0 \subseteq L^2(N_+) \oplus L^2(N_-)$ to the matrix valued function on $\mathrm{IC}(M, L^\infty(\mathbb{T}))$ determined by sending $(x,j) \in [0,1] \times \{0,1\}$ to

$$\sum_{p_0 \in I_q} \begin{pmatrix} K_j^{+,+}(p_0;x) f(p_0) & K_j^{-,+}(p_0;x) g(p_0) \\ K_j^{+,-}(p_0;x) g(p_0) & K_j^{-,-}(p_0;x) f(p_0) \end{pmatrix} p_0^2, \tag{4.19}$$

and $x \in \mu(q^{2\mathbb{Z}+1})$ to

$$\sum_{p_0 \in I_q} K_0^{+,+}(p_0;x) f(p_0) p_0^2 = \sum_{p_0 \in I_q} K_1^{+,+}(p_0;x) f(p_0) p_0^2.$$

Note that for $x = 0$, the matrix appearing in (4.19) is actually a direct sum of two matrix blocks after a basis transformation, since $W_{1,0}$ splits as a direct sum of four irreducible corepresentations, see [35, Proposition 10.13]. By Theorem 3.8.1, this map is unitary.

For completeness, we give the analogous result for the inverse Fourier transform $\mathcal{F}_2^{-1} : L^2(\hat{N}_+) \oplus L^2(\hat{N}_-) \to L^2(N_+) \oplus L^2(N_-)$. Since $\mathcal{Q}_0^{L^\infty(\mathbb{T})}$ is a restriction of the Plancherel transformation $\mathcal{Q}_L$, see Theorem 3.8.1, we see that $\mathcal{F}_2^{-1}$ can be considered as the restriction of $\mathcal{Q}_L^{-1}$. The transform $\mathcal{Q}_L^{-1}$ is described in Chapter 5 on the operator algebraic level. See also [92] for the algebraic counterpart. More explicitly, the spherical inverse Fourier transform is determined by

$$\mathcal{F}_2^{-1} : f \mapsto (\iota \otimes (f \cdot \hat{\varphi}))(W^*),$$

where $f \in \hat{N} \cap \mathfrak{n}_{\hat{\varphi}}$ is such that there is a normal functional on $\hat{M}$, denoted by $(f \cdot \hat{\varphi})$, which is determined by $(f \cdot \hat{\varphi})(x) = \hat{\varphi}(xf), x \in \mathfrak{n}_{\hat{\varphi}}^*$. By the decomposition of W (1.22), we find the following theorem.

Theorem 4.5.1. *For* $\sigma,\tau \in \{+,-\}$, *let* $f_1^{\sigma,\tau}, f_2^{\sigma,\tau} \in L^1([0,1]) \cap L^2([0,1])$, *where* $L^1([0,1])$ *and* $L^2([0,1])$ *should be understood with respect to the integral given by* $\int_{[0,1]} e(x)dx$. *Let* $f_d \in L^1(\mu(q^{2\mathbb{N}+1})) \cap L^2(\mu(q^{2\mathbb{N}+1}))$, *where* $L^1(\mu(q^{2\mathbb{N}+1}))$ *and* $L^2(\mu(q^{2\mathbb{N}+1}))$ *should be understood with respect to the integral* $\sum_{x\in\mu(q^{2\mathbb{N}+1})} e(x)$. *Define the function* $f \in L^2(\hat{N})$ *by*

$$f(x) = \begin{cases} \begin{pmatrix} f_1^{+,+}(x) & f_1^{-,+}(x) \\ f_1^{+,-}(x) & f_1^{-,-}(x) \end{pmatrix} \oplus \begin{pmatrix} f_2^{+,+}(x) & f_2^{-,+}(x) \\ f_2^{+,-}(x) & f_2^{-,-}(x) \end{pmatrix}, & x \in [0,1], \\ f_d(x), & x \in \mu(q^{2\mathbb{N}+1}). \end{cases}$$

Then,

$$\begin{aligned} &\int_{[0,1]} \left(f_1^{+,+}(x)\overline{K_1^{+,+}(p_0;x)} + f_1^{-,-}(x)\overline{K_1^{-,-}(p_0;x)}\right) e(x)dx \\ &+ \int_{[0,1]} \left(f_2^{+,+}(x)\overline{K_2^{+,+}(p_0;x)} + f_2^{-,-}(x)\overline{K_2^{-,-}(p_0;x)}\right) e(x)dx \\ &+ \sum_{x\in\mu(q^{2\mathbb{N}+1})} \left(f_d(x)\overline{K_1^{+,+}(p_0;x)} e(x)\right) \\ &\bigoplus \Bigg(\int_{[0,1]} \left(f_1^{-,+}(x)\overline{K_1^{+,-}(p_0;x)} + f_1^{+,-}(x)\overline{K_1^{-,+}(p_0;x)}\right) e(x)dx \\ &+ \int_{[0,1]} \left(f_2^{-,+}(x)\overline{K_2^{+,-}(p_0;x)} + f_2^{+,-}(x)\overline{K_2^{-,+}(p_0;x)}\right) e(x)dx \Bigg) u_0 \end{aligned} \tag{4.20}$$

exists for every $p_0 \in I_q$. *Moreover, (4.20) considered as a direct sum of functions in* p_0 *forms an element of* $L^2(N_+) \oplus L^2(N_-)$. *This mapping extends to a unitary map* $L^2(\hat{N}_+) \oplus L^2(\hat{N}_-) \to L^2(N_+) \oplus L^2(N_-)$, *which is inverse to* $\mathcal{F}_2$.

We explicitly state the formulae for the kernels $K_j^{\sigma,\tau}(p_0;x)$ for $x \in [-1,1] \cup \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1})$, which can be expressed in terms of little q-Jacobi functions. Using the notation of [35, Section 9] (or Section 1.8) for $S(\cdot)$ and $A(\cdot)$, we find by (1.24) and (1.25),

$$K_1^{+,+}(p_0;x) = S(-\lambda, p_0, p_0, 0) \times \begin{cases} 1 & p_0 < 0, \\ \frac{A(\lambda,1,0,+,+)}{A(\lambda,1,0,-,-)} & p_0 > 0. \end{cases}$$

$$K_2^{+,+}(p_0;x) = S(-\lambda, p_0, p_0, 0) \times \begin{cases} 1 & p_0 < 0, \\ -\frac{A(\lambda,1,0,+,+)}{A(\lambda,1,0,-,-)} & p_0 > 0. \end{cases}$$

$$K_1^{+,-}(p_0;x) = -S(\lambda, -p_0, p_0, 0) \times \begin{cases} \frac{A(\lambda,1,0,+,+)}{A(-\lambda,1,0,+,-)} & p_0 < 0, \\ -\frac{A(\lambda,1,0,+,+)}{A(-\lambda,1,0,-,+)} & p_0 > 0. \end{cases}$$

$$K_2^{+,-}(p_0;x) = -S(\lambda, -p_0, p_0, 0) \times \begin{cases} \frac{A(\lambda,1,0,+,+)}{A(-\lambda,1,0,+,-)} & p_0 < 0, \\ \frac{A(\lambda,1,0,+,+)}{A(-\lambda,1,0,-,+)} & p_0 > 0. \end{cases}$$

The fractions of the functions $A(\cdot)$ are phase factors. Here, $\lambda \in \mathbb{T}$ such that $\mu(\lambda) = x$. And, for $\lambda \in \mathbb{C}\backslash\{0\}$, simplifying (1.26),

$$\begin{aligned} & S(\pm\lambda, \mp p_0, p_0, 0) \\ =& |p_0|^2 \nu(p_0)^2 c_q^2 \sqrt{(\pm\kappa(p_0), -\kappa(p_0); q^2)_\infty} (\mp q^2; q^2)_\infty \quad \times \\ & \frac{(q^2, -q^2/\kappa(p_0), \lambda q^2, 1/\lambda, -q/\lambda; q^2)_\infty}{(\mathrm{sgn}(p_0)1/\lambda, \mathrm{sgn}(p_0)\lambda q^2, \pm q/\lambda; q^2)_\infty} \, {}_2\varphi_1 \left(\begin{matrix} -q/\lambda, -\lambda q \\ -q^2 \end{matrix} ; q^2, -q^2/\kappa(p_0) \right). \end{aligned}$$

The other kernels occuring in (4.19) can be expressed explicitly by means of the symmetry relation (4.18). The matrix coefficients $K_j^{\sigma,\tau}(p_0, x)$ are special types of little q-Jacobi functions, see also [35, Appendix B.5] and references given there.

4.6 Product formulae for little q-Jacobi functions

Let $N_{*,+}$ be the space of normal functionals in N_* which are zero on N_-. Let $N_{*,-}$ be the space of normal functionals in N_* which are zero on N_+. Note that $\varphi^\natural$ is a trace on N. Therefore every functional in $N_{*,+}$ is given by $f \cdot \varphi^\natural$, where f is a function on I_q so that $\sum_{p_0 \in I_q} f(p_0)p_0^2 < \infty$. Every functional in $N_{*,-}$ is given by $fu_0 \cdot \varphi^\natural$, where f is a function on $I_q \cap (-1,1)$ so that $\sum_{p_0 \in I_q \cap (-1,1)} f(p_0)p_0^2 < \infty$. So $N_* = N_{*,+} \oplus N_{*,-}$ as vector spaces.

Recall that δ_p denotes the function on I_q whose value is 1 in p and 0 elsewhere. We write $\delta_{p,+}$ for $\delta_p \cdot \varphi^\natural \in N_{*,+}$. For $p \in I_q$, we write $\delta_{p,-}$ for the functional $\delta_p u_0 \cdot \varphi^\natural \in N_{*,-}$. Only for $p \in I_q \cap (-1,1)$, this functional is non-zero.

Remark 4.6.1. Note that if one identifies N with the subalgebra $(1 \otimes L^\infty(I_q) \otimes 1) \cup (1 \otimes L^\infty(I_q)u_0 \otimes 1)$ of M acting on the GNS-space $\mathcal{K}$, then $\delta_{p,+} = \omega_{f_{m_0,p,t_0}, f_{m_0,p,t_0}}$ and $\delta_{p,-} = \omega_{f_{m_0,-p,t_0}, f_{m_0,p,t_0}}$, where $m_0 \in \mathbb{Z}, t_0 \in I_q$. This functional is independent of m_0 and t_0 if considered as a functional on N.

Using the fact that the Fourier transform preserves the $\mathbb{Z}_2$-gradings on N and $\hat{N}$, we see that for $p_1, p_2 \in I_q$:

$$\begin{aligned} &(\delta_{p_1,+} \otimes \delta_{p_2,+})\Delta^\natural \in N_{*,+}, & \qquad &(\delta_{p_1,-} \otimes \delta_{p_2,-})\Delta^\natural \in N_{*,+}, \\ &(\delta_{p_1,+} \otimes \delta_{p_2,-})\Delta^\natural \in N_{*,-}, & &(\delta_{p_1,-} \otimes \delta_{p_2,+})\Delta^\natural \in N_{*,-}. \end{aligned}$$

Hence, we see that for $f, g \in L^\infty(I_q)$, there exist constants $A_{p_0}(p_1,p_2), B_{p_0}(p_1,p_2)$, with $p_0, p_1, p_2 \in I_q$ and $C_{p_0}(p_1,p_2), D_{p_0}(p_1,p_2), p_0 \in I_q \cap (-1,1), p_1, p_2 \in I_q$ such that for any $p_1, p_2 \in I_q$ the four sums

$$\begin{aligned} &\textstyle\sum_{p_0 \in I_q} |A_{p_0}(p_1,p_2)|p_0^2, & &\textstyle\sum_{p_0 \in I_q} |B_{p_0}(p_1,p_2)|p_0^2, \\ &\textstyle\sum_{p_0 \in I_q \cap (-1,1)} |C_{p_0}(p_1,p_2)|p_0^2, & &\textstyle\sum_{p_0 \in I_q \cap (-1,1)} |D_{p_0}(p_1,p_2)|p_0^2, \end{aligned}$$

are finite and

$$\begin{aligned}
(\delta_{p_1,+} \otimes \delta_{p_2,+})\Delta^\natural(f) &= \sum_{p_0 \in I_q} A_{p_0}(p_1,p_2) f(p_0) p_0^2, & (4.21)\\
(\delta_{p_1,-} \otimes \delta_{p_2,-})\Delta^\natural(f) &= \sum_{p_0 \in I_q} B_{p_0}(p_1,p_2) f(p_0) p_0^2, & (4.22)\\
(\delta_{p_1,+} \otimes \delta_{p_2,-})\Delta^\natural(f u_0) &= \sum_{p_0 \in I_q \cap (-1,1)} C_{p_0}(p_1,p_2) f(p_0) p_0^2, & (4.23)\\
(\delta_{p_1,-} \otimes \delta_{p_2,+})\Delta^\natural(f u_0) &= \sum_{p_0 \in I_q \cap (-1,1)} D_{p_0}(p_1,p_2) f(p_0) p_0^2. & (4.24)
\end{aligned}$$

Putting $f = K_j^{\sigma,\tau}(p_0;x)$ for any $x \in [-1,0) \cup (0,1] \cup \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1}), j \in \{1,2\}$, this yields a product formula for ${}_2\varphi_1$-series. For $p_1, p_2 \in I_q$ and $p_3, p_4 \in I_q \cap (-1,1)$,

$$\begin{aligned}
K_j^{+,+}(p_1;x) K_j^{+,+}(p_2;x) &= \sum_{p_0 \in I_q} A_{p_0}(p_1,p_2) K_j^{+,+}(p_0;x) p_0^2, & (4.25)\\
K_j^{+,-}(p_3;x) K_j^{-,+}(p_4;x) &= \sum_{p_0 \in I_q} B_{p_0}(p_3,p_4) K_j^{-,-}(p_0;x) p_0^2, & (4.26)\\
K_j^{+,+}(p_1;x) K_j^{-,+}(p_3;x) &= \sum_{p_0 \in I_q \cap (-1,1)} C_{p_0}(p_1,p_3) K_j^{-,+}(p_0;x) p_0^2, & (4.27)\\
K_j^{-,+}(p_3;x) K_j^{-,-}(p_1;x) &= \sum_{p_0 \in I_q \cap (-1,1)} D_{p_0}(p_3,p_1) K_j^{-,+}(p_0;x) p_0^2. & (4.28)
\end{aligned}$$

Remark 4.6.2. Note that the gradings on N and $\hat{N}$ make that the left hand sides of (4.25) - (4.28) consists of a single product of two ${}_2\varphi_1$-functions.

Remark 4.6.3. If $x \in \mu(-q^{2\mathbb{N}+1} \cup q^{2\mathbb{N}+1})$, so that $W_{1,x}$ is a discrete series corepresentation, then equations (4.26) - (4.28) are trivial, i.e. they equate 0 to 0. If in addition $x < 0$, then (4.25) is also trivial.

Remark 4.6.4. From (4.21) – (4.24), we can also get formulae for the products

$$\begin{array}{ll}
K_j^{-,-}(p_1;x) K_j^{-,-}(p_2;x), & K_j^{+,-}(p_3;x) K_j^{-,+}(p_4;x),\\
K_j^{+,+}(p_1;x) K_j^{-,+}(p_3;x), & K_j^{-,+}(p_3;x) K_j^{-,-}(p_1;x),
\end{array}$$

where $j \in \{1,2\}$, $p_1, p_2 \in I_q$ and $p_3, p_4 \in I_q \cap (-1,1)$. However, using the symmetry (4.18), these formulae are already contained in (4.25) - (4.28).

In the remainder of this section we determine the coefficient functions A, B, C, and D. We explicitly show how to find A, the other coefficients can be found by the same method. Let $f = \delta_{p_0}$, so that the right hand side of (4.21) is equal to $A_{p_0}(p_1,p_2)p_0^2$. To determine the left hand side, note that $\delta_{p_0} = \delta_{\text{sgn}(p_0)}(e)\delta_{p_0^{-2}}(\gamma^*\gamma)$.

Recall that $\Delta(e) = e\otimes e \in N\otimes N$, so that $\Delta(\delta_{\mathrm{sgn}(p_0)}(e)) = \delta_{\mathrm{sgn}(p_0)}(e\otimes e) \in N\otimes N$. Hence,

$$\Delta^\natural(\delta_{\mathrm{sgn}(p_0)}(e)) = \Delta(\delta_{\mathrm{sgn}(p_0)}(e)) = \delta_1(e)\otimes\delta_{\mathrm{sgn}(p_0)}(e)+\delta_{-1}(e)\otimes\delta_{-\mathrm{sgn}(p_0)}(e) \in N\otimes N.$$

Note that by the relation $\Delta(x) = W^*(1\otimes x)W, x \in M$, we find that $\Delta(\delta_{p_0^{-2}}(\gamma^*\gamma))$ equals the projection onto the closure of

$$\mathrm{span}\left\{W^* f_{m',p',t'} \otimes f_{m,p_0,t} \mid m, m' \in \mathbb{Z}, p', t, t' \in I_q\right\}.$$

This projection is given by the formula

$$\mathcal{K}\otimes\mathcal{K} \to \mathcal{K}\otimes\mathcal{K} : v \mapsto \sum_{m,m'\in\mathbb{Z},p',t,t'\in I_q} \langle v, W^* f_{m',p',t'}\otimes f_{m,p_0,t}\rangle W^* f_{m',p',t'}\otimes f_{m,p_0,t},$$

where the sum is norm convergent. Note that for $x \in N$, $\Delta(x) \in M^\gamma \otimes M^\beta$, so that $\Delta^\natural(x) = (\iota \otimes T_\beta T_\gamma)\Delta(x) = (\iota \otimes (T^+ + T^-))\Delta(x)$. Since $\Delta^\natural(\delta_{p_0^{-2}}(\gamma^*\gamma)) \in N \otimes N$, we find that for any $m_1, m_2 \in \mathbb{Z}$ and $t_1, t_2 \in I_q$,

$$\begin{aligned}
&(\delta_{p_1} \otimes \delta_{p_2})\Delta^\natural(\delta_{p_0^{-2}}(\gamma^*\gamma))\\
=&\langle(\iota \otimes (T^+ + T^-))(\Delta(\delta_{p_0^{-2}}(\gamma^*\gamma)))f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}, f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}\rangle\\
=&\langle\Delta(\delta_{p_0^{-2}}(\gamma^*\gamma))f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}, f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}\rangle\\
=&\sum_{m',m\in\mathbb{Z},p',t,t'\in I_q} |\langle f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}, W^* f_{m',p',t'} \otimes f_{m,p_0,t}\rangle|^2,
\end{aligned} \tag{4.29}$$

where the equations follow from Remark 4.6.1 and Proposition 4.3.2, the definition of T^+ and T^- and the discussion above. Recall the functions $a_p(x, y), p, x, y \in I_q$ from [35, Definition 6.2]. From [51, Proposition 4.5 and 4.10], we see that (4.29) equals

$$\begin{aligned}
&\sum_{t\in I_q} \left(\frac{t}{t_2}\right)^2 a_t(\mathrm{sgn}(p_0p_2t_2)p_1q^{m_2}t, t_2)^2 a_{p_0}(p_1, p_2)^2\\
=&\sum_{t\in I_q} a_{t_2}(\mathrm{sgn}(t_2)|p_1|q^{m_2}t, t)^2 a_{p_0}(p_1, p_2)^2 = a_{p_0}(p_1, p_2)^2.
\end{aligned} \tag{4.30}$$

Here, the first equality follows from [35, Eqn. (24)] and the fact that by definition $a_{p_0}(p_1, p_2)^2 = 0$ if $\mathrm{sgn}(p_0p_2) \neq \mathrm{sgn}(p_1)$. The last equality follows from [35, Proposition 6.3]. Hence, using [75, Section IX.4, Eqn. (4)] in the second equality,

$$\begin{aligned}
A_{p_0}(p_1, p_2)p_0^2 =&(\delta_{p_1} \otimes \delta_{p_2})\Delta^\natural(\delta_{\mathrm{sgn}(p_0)}(e)\delta_{p_0^{-2}}(\gamma^*\gamma))\\
=&(\delta_{p_1} \otimes \delta_{p_2})(\iota \otimes T_\gamma)\left(\delta_{\mathrm{sgn}(p_0)}(\Delta(e))\delta_{p_0^{-2}}\Delta(\gamma^*\gamma)\right)\\
=&(\delta_{p_1} \otimes \delta_{p_2})\delta_{\mathrm{sgn}(p_0)}(\Delta(e))(\iota \otimes T_\gamma)\left(\delta_{p_0^{-2}}\Delta(\gamma^*\gamma)\right)\\
=&\delta_{\mathrm{sgn}(p_0),\mathrm{sgn}(p_1p_2)}a_{p_0}(p_1, p_2)^2 = a_{p_0}(p_1, p_2)^2.
\end{aligned}$$

Recall [51, Definition 3.1] that by definition $a_{p_0}(p_1,p_2) = 0$ if $\delta_{\text{sgn}(p_0),\text{sgn}(p_1p_2)} = 0$, so indeed the last equality follows. By a similar computation we get,

$$\begin{aligned}
B_{p_0}(p_1,p_2)p_0^2 &= a_{p_0}(p_1,p_2)a_{p_0}(-p_1,-p_2),\\
C_{p_0}(p_1,p_2)p_0^2 &= a_{p_0}(p_1,-p_2)a_{-p_0}(p_1,p_2),\\
D_{p_0}(p_1,p_2)p_0^2 &= a_{p_0}(-p_1,p_2)a_{-p_0}(p_1,p_2).
\end{aligned}$$

Remark 4.6.5. In particular, the sums in (4.25) – (4.28) run only through either the positive or the negative numbers, since $a_z(x,y) = 0$ if sgn$(xyz) = -1$.

Remark 4.6.6. It is also possible to obtain (4.21) - (4.24) from [35, Proposition 4.10], using the pairing between M and $\{\lambda(\omega) \mid \omega \in M_*\} \subseteq \hat{M}$, defined by $\langle x, \lambda(\omega)\rangle = \omega(x)$.

Remark 4.6.7. In case of the group $SU(1,1)$ we know [94, Chapter 6] that there exists an addition formula corresponding to the product formula, i.e. the product formula corresponds to the constant term in the addition formula. It would be of interest to obtain addition formulae corresponding to (4.25) - (4.28).

Chapter 5

L^p-Fourier theory

The Fourier transform is one of the most powerful tools coming from abstract harmonic analysis. Many classical applications, in particular in the direction of L^p-spaces, can be found in for example [34]. Here we extend this tool by giving a definition of a Fourier transform on the non-commutative L^p-spaces associated with a locally compact quantum group. This gives a link between quantum groups and non-commutative measure theory.

Recall that the Fourier transform on locally compact abelian groups can be defined in an L^p-setting for p any real number between 1 and 2. This is done in the following way. Let G be a locally compact abelian group and let $\hat{G}$ be its Pontrjagin dual. For a L^1-function f on G, we define its Fourier transform $\hat{f}$ to be the function on $\hat{G}$, which is defined by

$$\hat{f}(\pi) = \int f(x)\pi(x)d_l x, \qquad \pi \in \hat{G}. \tag{5.1}$$

Then $\hat{f}$ is a continuous function on $\hat{G}$ vanishing at infinity. So we can consider this transform as a bounded map $\mathcal{F}_1 : L^1(G) \to L^\infty(\hat{G})$. The Plancherel theorem yields that if f is moreover a L^2-function on G, then $\hat{f}$ is a L^2-function on $\hat{G}$ and this map extends to a unitary map $\mathcal{F}_2 : L^2(G) \to L^2(\hat{G})$.

It is known that the Fourier transform can be generalized in a L^p-setting by means of the Riesz-Thorin theorem, see [3]. The statement of this theorem directly implies the following. For any p, with $1 \leq p \leq 2$, the linear map $L^1(G) \cap L^2(G) \to L^2(\hat{G}) \cap L^\infty(\hat{G}) : f \mapsto \hat{f}$ extends uniquely to a bounded map

$$\mathcal{F}_p : L^p(G) \to L^q(\hat{G}), \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

This map $\mathcal{F}_p$ is known as the L^p-Fourier transform.

Since we have a von Neumann algebraic interpretation for quantum groups at hand, it is natural to ask if the L^p-Fourier transform can be defined in this context. This is for two reasons. First of all, this framework studies quantum groups in a measurable setting which appeals to a more general interest: what links can be found between on the one hand non-commutative measure spaces, in particular non-commutative L^p-spaces, and on the other hand the theory of quantum groups. The L^p-Fourier transform studied in the present chapter establishes such a link. Secondly, the existence of a Pontrjagin dual is always guaranteed in the Kustermans-Vaes setting. This is an essential ingredient for defining Fourier transforms.

The L^1- and L^2-Fourier transform already appear in the present theory of quantum groups. In fact, they are implicitly used to define duals of quantum groups. Let us comment on this.

First of all, the L^2-Fourier transform is implicitly used in the construction of the Pontrjagin dual of a (von Neumann algebraic) quantum group. For the classical case of a locally compact abelian group G, let $(L^\infty(G), \Delta_G)$ be the usual quantum group associated with it. Its dual is given by $(\mathcal{L}(G), \hat{\Delta}_G)$, where $\mathcal{L}(G)$ is the group von Neumann algebra of G. This structure is spatially isomorphic to $(L^\infty(\hat{G}), \Delta_{\hat{G}})$ be means of the L^2-Fourier transform. That is $L^\infty(\hat{G}) = \mathcal{F}_2 \mathcal{L}(G) \mathcal{F}_2^{-1}$ and similarly the coproduct and other concepts translate, see Example 1.2.4.

Secondly, Van Daele [92] explains how the Fourier transform should be interpreted on the algebraic level of quantum groups. In his concluding remarks he suggests to study this transform in the operator algebraic framework. This investigation is carried out here. We take Van Daele's definition, which agrees with the classical transform (5.1), as a starting point for defining a L^2-Fourier transform in the operator algebraic framework.

Finally, an operator algebraic interpretation of the Fourier transform can be found in [43]. The main ideas for our L^2-Fourier transform first appear here. However, the suggested Fourier transform [43, Definition 3] is well-defined only if the Haar weights of a quantum group are states, i.e. if the quantum group is compact. In the more general situation one has to give a more careful definition, which we work out in Section 5.6.

The present chapter is related to a collection of papers studying module structures of L^p-spaces associated to the Fourier algebra of locally compact groups [17], [18], [31]. These papers are based on the theory of non-commutative L^p-spaces associated to arbitrary (not necessarily semi-finite) von Neumann algebras, which we recall below.

When dealing with these spaces, we are confronted with the following obstruction. For classical L^p-spaces associated with a measure space X, there is a clear understanding of the the intersections of L^p-spaces by means of disjunction of sets. So $L^p(X) \cap L^{p'}(X)$ gives the intersection of $L^p(X)$ and $L^{p'}(X)$. For non-commutative L^p-spaces it is more difficult to find the intersection of two such spaces. In fact, there is a choice which determines the intersection and which depends on a *complex interpolation parameter* $z \in \mathbb{C}$. In [31] the parameter $z = -1/2$ is used, whereas [17] focuses on the case $z = 0$ in order to define module actions. In the final remarks of [18], it is questioned which parameter would fit best for quantum groups.

One of the results of the present chapter is that to define a L^p-Fourier transform, one is obliged to choose the parameter $z = -1/2$. We also determine intersections of the L^1- and L^2-space and of the L^2- and L^∞-space associated with a von Neumann algebra for this parameter, which are natural spaces.

We explain the structure of this chapter. Sections 5.1 and 5.2 are preliminary. We first recall the complex interpolation method and next, we show how it was used by Izumi (building on the earlier work by Kosaki [54] and Terp [78]) to define non-commutative L^p-spaces. At this point we also introduce the complex interpolation parameter $z \in \mathbb{C}$.

In Section 5.3 we specialize the theory of Izumi's non-commutative L^p-spaces for the particular complex interpolation parameter $z = -1/2$. It allows us to introduce short hand notation.

Next, we recall the L^p-spaces as defined by Hilsum [39], see Section 5.4. Our focus here is a proposition that allows us to develop the theory in a much more concrete and computable way.

As indicated the study of the intersections of L^p-spaces becomes more intricate in the non-commutative setting. In Section 5.5 we determine the intersections of L^1- and L^2-space and of the L^2- and L^∞-space associated with a von Neumann algebra. These intersections turn out to be well-known spaces in the theory of quantum groups. This gives a confirmation that our choice for the interpolation parameter made at the beginning is a natural one. Moreover, it gives the necessary ammunition to apply the re-iteration theorem. We warn the reader that the contents of Section 5.5 are relatively technical and if one is more interested in Fourier theory on quantum groups, one can skip Section 5.5 at first reading.

In Section 5.6 we define the L^p-Fourier transform. We start with the L^1- and L^2-theory and then obtain the L^p-Fourier transform ($1 \leq p \leq 2$) through the complex interpolation method, a method similar to the Riesz-Thorin theorem mentioned before.

In Section 5.7 we define a convolution product in the L^p-setting and show that the Fourier transform turns the convolution product into a product.

In Section 5.8 is devoted to the 'right' side of the story. We find a 'right' Fourier transform. Furthermore, we show that there exists a pairing between the L^p-space associated with M and the L^p-space associated with $\hat{M}$ for $1 \leq p \leq 2$. The pairing extends the pairing described in [43].

We mention that in Sections 5.4 - 5.7 we mainly work with the complex interpolation parameter $z = -\frac{1}{2}$. In the final section, we return to the general theory. Section 5.9 is devoted to a theorem that proves that the interpolation parameter used in earlier sections is distinguished if one likes to define a Fourier theory in the L^p-setting. That is, we prove that given the L^1-Fourier transform, there is only one choice for the interpolation parameter that allows an L^p-Fourier transform. This justifies our choice for this parameter made in the beginning.

The contents of this chapter, except for Section 5.8 is contained in the forthcoming paper [7]. Section 5.8 has been added later for completeness.

5.1 The complex interpolation method

We recall the complex interpolation method as explained in [3, Section 4.1].

Definition 5.1.1. Let E_0, E_1 be Banach spaces. The couple (E_0, E_1) is called a *compatible couple* (of Banach spaces) if E_0 and E_1 are continuously embedded into a Banach space E.

Note that we suppress E in the notation (E_0, E_1). We can consider the spaces $E_0 \cap E_1$ and $E_0 + E_1$ interpreted within E and equip them with norms

$$\begin{aligned} \|x\|_{E_0\cap E_1} &= \max\{\|x\|_{E_0}, \|x\|_{E_1}\}, && x \in E_0 \cap E_1, \\ \|x\|_{E_0+E_1} &= \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} \mid x_0 + x_1 = x\}, && x \in E_0 + E_1, \end{aligned}$$

which make them Banach spaces. In that case we can consider E_0 and E_1 as subspaces of $E_0 + E_1$.

Definition 5.1.2. A *morphism between compatible couples* (E_0, E_1) and (F_0, F_1) is a bounded map $T : E_0+E_1 \to F_0+F_1$ such that for any $j \in \{0, 1\}$, $T(E_j) \subseteq F_j$ and the restriction $T : E_j \to F_j$ is bounded.

Remark 5.1.3. Let (E_0, E_1) and (F_0, F_1) be compatible couples. If $T_0 : E_0 \to F_0, T_1 : E_1 \to F_1$ are bounded maps such that T_0 and T_1 agree on $E_0 \cap E_1$, then we call T_0 and T_1 *compatible morphisms*. In this case, there is a unique bounded map $T : E_0 + E_1 \to F_0 + F_1$. This gives a way to find morphisms of compatible couples.

Now we describe the complex interpolation method. Let (E_0, E_1) be a compatible couple. Let $\mathcal{S} = \{z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le 1\}$ and let $\mathcal{S}^\circ$ denote its interior. Let $\mathcal{G}(E_0, E_1)$ be the set of functions $f : \mathcal{S} \to E_0 + E_1$ such that

(1) f is bounded and continuous on $\mathcal{S}$ and analytic on $\mathcal{S}^\circ$;

(2) For $t \in \mathbb{R}, j \in \{0, 1\}$, $f(it + j) \in E_j$ and $t \mapsto f(it + j)$ is continuous and bounded with respect to the norm on E_j;

(3) For $j \in \{0, 1\}$, $\|f(it + j)\|_{E_j} \to 0$ as $t \to \infty$.

Note that at this point our notation is different from [3] and [40], where $\mathcal{G}$ is denoted by $\mathcal{F}$, which we reserve for the Fourier transform. For $f \in \mathcal{G}(E_0, E_1)$, we define a norm

$$|||f||| = \max\{\sup \|f(it)\|_{E_0}, \sup \|f(it + 1)\|_{E_1}\}.$$

Let $\theta \in [0,1]$. We define $(E_0, E_1)_{[\theta]} \subseteq E$ to be the space $\{f(\theta) \mid f \in \mathcal{G}(E_0, E_1)\}$ with norm

$$\|x\|_{[\theta]} = \inf\{\|\!|f|\!\| \mid f(\theta) = x, f \in \mathcal{G}(E_0, E_1)\}.$$

With this norm, $(E_0, E_1)_{[\theta]}$ is a Banach space [3, Theorem 4.1.2].

Definition 5.1.4. The assignment from compatible couples of Banach spaces to Banach spaces which is given by $C_\theta : (E_0, E_1) \to (E_0, E_1)_{[\theta]}$ is called the *complex interpolation method* (at parameter $\theta \in [0,1]$). $(E_0, E_1)_{[\theta]}$ is called a complex interpolation space.

The following Riesz-Thorin-like theorem plays a central role in the present chapter. It gives the functorial property of the complex interpolation method.

Theorem 5.1.5 (Theorem 4.1.2 of [3])**.** *Let $\theta \in [0,1]$. Let T be a morphism between compatible couples (E_0, E_1) and (F_0, F_1). Then, it restricts to a bounded linear map $T : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$. The norm is bounded by $\|T\| \leq \|T : E_0 \to F_0\|^{1-\theta} \|T : E_1 \to F_1\|^{\theta}$.*

If we let C_θ of Definition 5.1.4 act on the morphisms $T : (E_0, E_1) \to (F_0, F_1)$ of compatible couples by assigning its restriction $T : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ to it, we see that C_θ is a functor.

Remark 5.1.6. Using the notation of Remark 5.1.3, the compatible morphisms T_0, T_1 give rise to a morphism $C_\theta(T) : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ on the interpolation spaces with norm $\|C_\theta(T)\| \leq \|T_0\|^{1-\theta} \|T_1\|^{\theta}$.

We also need the following useful fact.

Lemma 5.1.7 (Theorem 4.2.2 of [3])**.** *Let (E_0, E_1) be a compatible couple and $\theta \in [0,1]$. $E_0 \cap E_1$ is dense in $(E_0, E_1)_{[\theta]}$.*

5.2 Non-commutative L^p-spaces

A non-commutative L^p-space can be associated to any von Neumann algebra M. In fact there are many ways to do this. If M is semi-finite, i.e. it admits a normal semi-finite faithful trace τ, then one can define $L^p(M)$ as the space of closed densely defined operators x affiliated with M for which

$$\|x\|_p := \left(\sup_{n \in \mathbb{N}} \tau \left(\int_{[0,n]} \lambda^p dE_\lambda \right) \right)^{1/p} < \infty,$$

where $|x| = \int_{[0,\infty)} \lambda dE_\lambda$ is the spectral decomposition of $|x|$. If $M = L^\infty(X)$, we recover the classical spaces $L^p(X)$ for a certain measure space X.

Since the introduction of Tomita-Takesaki theory L^p-spaces have been defined for not necessarily semi-finite von Neumann algebras. Definitions of non-commutative L^p-spaces have been given by Haagerup [37], [77], Hilsum [39], Terp [78] and Izumi [40]. The definitions can be shown te be equivalent. That is, the L^p-spaces obtained by the various definitions are isometrically isomorphic Banach spaces. For a good introduction to this theory we refer to [77], where a comparison of Haagerup's definition and Hilsum's definition is made.

Here we mainly use Izumi's definition [40] which is abstract in nature. Izumi defines L^p-spaces associated with M by means of the complex interpolation method; a method that admits a property that is reminiscent of the Riesz-Thorin theorem, see the introduction of this chapter. It is for this reason that Izumi's definition is the most suitable context to work in.

A drawback of Izumi's approach is that the more concrete approach of the other definitions is absent. Whenever it feels appropriate we comment on this.

Let us describe the origin of Izumi's abstract definition. In [78] Terp shows that the L^p-spaces as introduced by Hilsum can be obtained by applying the complex interpolation method to a specific compatible couple (M, M_*), see [78, Theorem 36]. Izumi [40] realized, greatly inspired by the ideas by Kosaki [54], that there is more than one way to turn (M, M_*) into a compatible couple in order to obtain the L^p-spaces through interpolation. His idea is to *define* non-commutative L^p-spaces as complex interpolation spaces of certain compatible structures. It is this definition which we recall here.

Initially, we present the general picture. However, in the larger part of the present chapter, we only work with the complex interpolation parameters $z = -1/2$ and $z = 1/2$ (we introduce the parameter in a minute). We specialize the theory for these parameters in Sections 5.3 and 5.4 and introduce short hand notation there. The more general theory is used in Section 5.9, where we prove that there is in principle only one interpolation parameter that allows a L^p-Fourier transform, namely $z = -1/2$.

Fix a von Neumann algebra M with normal semi-finite faithful weight φ. Let ∇ and J be the modular operator and conjugation for φ, see Appendix A.2. The following construction of L^p-spaces can be found in [40].

Definition 5.2.1. For $z \in \mathbb{C}$, we put

$$\begin{aligned} L_{(z)} = \{x \in M \mid \exists \varphi_x^{(z)} \in M_* \text{ such that } \forall a, b \in \mathcal{T}_\varphi : \\ \varphi_x^{(z)}(a^*b) = \langle xJ\nabla^{\bar{z}}\Lambda(a) \mid J\nabla^{-z}\Lambda(b)\rangle \Big\}. \end{aligned}$$

The number $z \in \mathbb{C}$ will be called the *complex interpolation parameter*.

Remark 5.2.2. We will mainly be dealing with the cases $z = -1/2$ and $z = 1/2$. Note that if φ is a state, then for any $x \in M$, we see that for $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned} &\langle xJ\nabla^{-1/2}\Lambda(a) \mid J\nabla^{1/2}\Lambda(b)\rangle \\ =&\langle xJ\nabla^{1/2}\Lambda(\sigma_i(a)) \mid J\nabla^{1/2}\Lambda(b)\rangle = \varphi(bx\sigma_{-i}(a^*)) = \varphi(a^*bx), \end{aligned}$$

and hence $L_{(-1/2)} = M$ and $\varphi_x^{(-1/2)} = x\varphi$. Similarly, $L_{(1/2)} = M$ and $\varphi_x^{(1/2)} = \varphi x$.

The following proposition implies that there are plenty of elements contained in $L_{(z)}$.

Proposition 5.2.3 (Propostion 2.3 of [40]). *$\mathcal{T}_\varphi^2 = \{ab \mid a, b \in \mathcal{T}_\varphi\}$ is contained in $L_{(z)}$.*

We are now able to construct Izumi's L^p-spaces using the complex interpolation method. First, we define a compatible couple. For $x \in L_{(z)}$ we define a norm:

$$\|x\|_{L_{(z)}} = \max\{\|x\|, \|\varphi_x^{(z)}\|\}.$$

We define norm-decreasing injections:

$$i^{\infty}_{(z)} : L_{(z)} \to M : x \mapsto x; \qquad i^{1}_{(z)} : L_{(z)} \to M_* : x \mapsto \varphi_x^{(z)}.$$

Using the duals of the maps, we obtain the following diagram. Note that $(i^{\infty}_{(-z)})^* : M^* \to L^*_{(-z)}$ is restricted to M_*.

$$\begin{tikzcd}
& M_* \arrow[dr, hook, "(i^{\infty}_{(-z)})^*"] & \\
L_{(z)} \arrow[ur, hook, "i^{1}_{(z)}"] \arrow[r, hook, "i^{p}_{(z)}"] \arrow[dr, hook, "i^{\infty}_{(z)}"'] & L^p_{(z)}(M) \arrow[r, hook] & L^*_{(-z)}. \\
& M \arrow[ur, hook, "(i^{1}_{(-z)})^*"'] &
\end{tikzcd} \tag{5.2}$$

Now [40, Theorem 2.5] yields that the outer rectangle of (5.2) commutes. This turns (M, M_*) into a compatible couple of Banach spaces.

Definition 5.2.4. For $p \in (1, \infty)$, we define $L^p_{(z)}(M)$ to be the complex interpolation space $(M, M_*)_{[1/p]}$. We set $L^1_{(z)}(M) = M_*$ and $L^{\infty}_{(z)}(M) = M$.

Remark 5.2.5. In fact, $(M, M_*)_{[1]}$ is isometrically isomorphic to M_*. Indeed, interpreting every space within $L^*_{(-z)}$, we see that by definition $(M, M_*)_{[1]}$ is a closed subspace of M_* with respect to the norm of M_*. By Lemma 5.1.7, the space $L_{(z)}$ is dense in $(M, M_*)_{[1]}$. Furthermore, by Proposition 5.2.3 we see that $\mathcal{T}_\varphi^2$ is contained in $L_{(z)}$ and it is a straightforward check to see that for $a, b \in \mathcal{T}_\varphi$ the functional $i^1_{(z)}(ab)$ is given by $M \ni x \mapsto \varphi(\sigma_{iz+i/2}(b)x\sigma_{-i(\bar{z}-1/2)}(a)^*)$. Functionals of this form are dense in M_*. Hence, the claim follows.

Since for our purposes it is easier to identify the L^1-space with M_*, we made this identification straightaway.

However, it is not true in general that $(M, M_*)_{[0]} = M$. By Lemma 5.1.7, we see that $(M, M_*)_{[0]}$ equals the norm closure of $L_{(z)}$ in M. We see in Theorem 5.3.1 that $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$ and the closure of $\mathfrak{n}_\varphi$ in M does generally not equal M.

By Lemma 5.1.7, $L_{(z)}$ can be embedded in $L^p_{(z)}(M)$. This map is denoted by $\iota^p_{(z)}$. Note that by definition of the complex interplation method $L^p_{(z)}(M)$ is a linear subspace of $L^*_{(-z)}$.

Notation 5.2.6. The map $i^{\infty}_{(z)} : L_{(z)} \to M$ is basically the inclusion of a subspace. Therefore, it is convenient to omit the map $i^{\infty}_{(z)}$ in our notation if the norms of the spaces do not play a role in the statement. Similarly, we do not introduce notation for the inclusion of $L^p_{(z)}(M)$ in $L^*_{(-z)}$, where $p \in (1, \infty)$.

A priori one could think that $L^p_{(z)}(M)$ and $L^p_{(z')}(M)$ with $z \neq z'$, are different as Banach spaces. However, Izumi proves that they are isometrically isomorphic. Terp [78] considers the case $z = 0$. The main result of [78] is that $L^p_{(0)}(M)$ is isometrically isomorphic to the L^p-spaces by Hilsum [39]. We come back to this in Section 5.4.

Theorem 5.2.7 (Theorem 3.8 of [40]). *For $z, z' \in \mathbb{C}$, there is an isometric isomorphism*

$$U_{p,(z',z)} : L^p_{(z)}(M) \to L^p_{(z')}(M), \qquad p \in (1, \infty),$$

such that for $a \in \mathcal{T}^2_{\varphi}$,

$$U_{p,(z',z)}(i^p_{(z)}(a)) = i^p_{(z')}(\sigma_{i\frac{r'-r}{p} - (s'-s)}(a)), \tag{5.3}$$

where $z = r + is$ and $z' = r' + is'$, $r, r', s, s' \in \mathbb{R}$.

Although the L^p-spaces appearing in (5.2) are isomorphic for different complex interpolation parameters, the intersections $L^p_{(z)}(M) \cap L^{p'}_{(z)}(M)$ defined by this figure does depend on z. In any case, by [40, Corollary 2.13],

$$(i^1_{(-z)})^*(L_{(z)}) = (i^1_{(-z)})^*(M) \cap (i^{\infty}_{(-z)})^*(M_*), \tag{5.4}$$

i.e. if one consideres $L_{(z)}$, M, M_* as subspaces of $L^*_{(-z)}$, then $L_{(z)} = M \cap M_*$.

Remark 5.2.8. This section would not be complete without mentioning that interpolation properties of non-commutative L^p-spaces in case φ is a state have already been considered by Kosaki [54].

5.3 Specializations for the complex interpolation parameters

In the rest of this chapter we mainly work with the parameter $z = -1/2$. In order to study these spaces also the parameter $z = 1/2$ plays a role. In this section we specialize the theory for these parameters. The following proposition shows that $L_{(-1/2)}$ and $L_{(1/2)}$ can be described by a condition that is in general easier to check. If φ is a state it reduces to Remark 5.2.2.

Proposition 5.3.1. *We have the following alternative descriptions:*

1. *Let* $L = \{x \in \mathfrak{n}_{\varphi} \mid \exists\, {}_x\varphi \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_{\varphi} : \ {}_x\varphi(y^*) = \varphi(y^* x)\}$. *Then* $L = L_{(-1/2)}$.

2. *Let* $R = \{x \in \mathfrak{n}_\varphi^* \mid \exists \varphi_x \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_\varphi : \ \varphi_x(y) = \varphi(xy)\}$. *Then* $R = L_{(1/2)}$.

Proof. We only give the proof of (1), since (2) can be proved similarly. We first prove $\subseteq$. For $x \in L$, $a, b \in \mathcal{T}_\varphi$,

$$
\begin{aligned}
&{}_x\varphi(a^*b) = \varphi(a^*bx) = \varphi(bx\sigma_{-i}(a^*)) = \langle x\Lambda(\sigma_{-i}(a^*)), \Lambda(b^*)\rangle \\
=&\langle x\nabla J\nabla^{1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle = \langle xJ\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle.
\end{aligned}
$$

Hence $x \in L_{(-1/2)}$ and ${}_x\varphi = \varphi_x^{(-1/2)}$.

To prove $\supseteq$ we first prove that $M\mathcal{T}_\varphi^2 \subseteq L_{(-1/2)}$. Indeed, let $x \in M$ and let $c, d \in \mathcal{T}_\varphi$. The functional $M \ni y \mapsto \varphi(\sigma_i(d)yxc)$ is normal. Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$
\begin{aligned}
&\langle xcdJ\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle = \langle \Lambda(xcd\sigma_{-i}(a^*)), \Lambda(b^*)\rangle \\
=&\varphi(bxcd\sigma_{-i}(a^*)) = \varphi(\sigma_i(d)a^*bxc).
\end{aligned}
$$

Hence, $xcd \in L_{(-1/2)}$.

Next, we prove that $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$. Take $x \in L_{(-1/2)}$ and let $(e_j)_{j\in J}$ be a bounded net in $\mathcal{T}_\varphi$ such that $\sigma_i(e_j)$ is bounded and such that $e_j \to 1$ σ-weakly, see [78, Lemma 9] or Lemma A.6.2. Then, $xe_j \to x$ σ-weakly. Furthermore,

$$
\|\Lambda(xe_j)\|^2 = \varphi(e_j^*x^*xe_j) = \varphi^{(-1/2)}_{xe_j\sigma_{-i}(e_j^*)}(x^*) \leq \|\varphi^{(-1/2)}_{xe_j\sigma_{-i}(e_j^*)}\|\|x\|, \tag{5.5}
$$

where the second equality is due to the previous paragraph. By [40, Proposition 2.6],

$$
\varphi^{(-1/2)}_{xe_j\sigma_{-i}(e_j^*)} = \varphi_x^{(-1/2)}\sigma_i(e_j)e_j^*, \tag{5.6}
$$

where for $\omega \in M_*, y \in M$, ωy is the normal functional defined by $(\omega y)(a) = \omega(ya), a \in M$. From (5.5) and (5.6) it follows that $(\Lambda(xe_j))_{j\in J}$ is a bounded net. Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$
\langle \Lambda(xe_j), \Lambda(ab)\rangle = \varphi(b^*a^*xe_j) = \varphi(a^*xe_j\sigma_{-i}(b^*)) \to \varphi(a^*x\sigma_{-i}(b^*)).
$$

Since $(\Lambda(xe_j))_{j\in J}$ is bounded, this proves that $(\Lambda(xe_j))_{j\in J}$ is weakly convergent. Since Λ is σ-weak/weak closed, this implies that $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$. So $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$.

To finish the proof, let again $x \in L_{(-1/2)}$ and let $a, b \in \mathcal{T}_\varphi$. We prove that $\varphi_x^{(-1/2)}((ab)^*) = \langle \Lambda(x), \Lambda(ab)\rangle$. The proposition then follows, since $\mathcal{T}_\varphi^2$ is a σ-weak/weak-core for Λ by Lemma A.6.3. The proposition follows from:

$$
\begin{aligned}
&\langle \Lambda(x), \Lambda(ab)\rangle = \varphi(b^*a^*x) = \varphi(a^*x\sigma_{-i}(b^*)) \\
=&\langle xJ\nabla^{-1/2}\Lambda(b), J\nabla^{1/2}\Lambda(a^*)\rangle = \varphi_x^{(-1/2)}(b^*a^*).
\end{aligned}
$$

□

In particular, it follows from Proposition 5.3.1 that for $y \in \mathfrak{n}_\varphi$,

$$
\begin{aligned}
{}_x\varphi(y^*) = &\quad \varphi(y^*x), && x \in L,\\
\varphi_x(y) = &\quad \varphi(xy), && x \in R.
\end{aligned}
\tag{5.7}
$$

We emphasize that one has to be careful that (5.7) does not make sense for every $x, y \in M$. Also, (5.7) justifies why (5.2) is also called the *left injection* for $z = -1/2$ and the *right injection* for $z = 1/2$.

Part of Corollary 5.3.2 is already proved in [40]. Using the alternative descriptions of Proposition 5.3.1, it is easy to prove the remaining statements.

Corollary 5.3.2. *We have inclusions $M\mathcal{T}_\varphi^2 \subseteq L$, $\mathcal{T}_\varphi^2 M \subseteq R$, $\mathcal{T}_\varphi^2 \subseteq L \cap R$, $L\mathcal{T}_\varphi \subseteq L$, $\mathcal{T}_\varphi R \subseteq R$, $ML \subseteq L$ and $RM \subseteq R$. Moreover, $R = \{x^* \mid x \in L\}$ and for $x \in L$, $\varphi_{x^*} = \overline{{}_x\varphi}$.*

Proof. The first inclusion has already been proved in the proof of Proposition 5.3.1. Here we have proved that for $x \in M, a, b \in \mathcal{T}_\varphi$, ${}_{xab}\varphi(z) = \varphi(\sigma_i(b)zxa), z \in M$. Similarly, one can prove that for $x, z \in M, a, b \in \mathcal{T}_\varphi, y_l \in L, y_r \in R$,

$$
\begin{array}{lll}
\varphi_{abx}(z) = \varphi(bx\, z\, \sigma_{-i}(a)); & \varphi_{ab}(z) = \varphi(b\, z\, \sigma_{-i}(a)); & {}_{ab}\varphi(z) = \varphi(\sigma_i(b)\, z\, a);\\
{}_{y_l a}\varphi(z) = {}_{y_l}\, \varphi(\sigma_i(a)z); & \varphi_{ay_r}(z) = \varphi_{y_r}^{(1/2)}(z\sigma_{-i}(a)); & {}_{xy_l}\varphi(z) = {}_{y_l}\, \varphi(zx);\\
\varphi_{y_r x}(z) = \varphi_{y_r}(xz); & \varphi_{x^*} = \overline{{}_x\varphi}. &
\end{array}
$$

□

Since we are mainly dealing with complex interpolation parameter $z = -1/2$ and $z = 1/2$, it is more convenient to adapt our notation.

Notation 5.3.3. We use the following short hand notations. For $p \in [1, \infty]$,

$$
\begin{array}{llll}
L^p(M)_{\mathrm{left}} = L^p_{(-1/2)}(M), & L = L_{(-1/2)}, & l^p = i^p_{(-1/2)}, & {}_x\varphi = \varphi_x^{(-1/2)} \text{ for } x \in L,\\
L^p(M)_{\mathrm{right}} = L^p_{(1/2)}(M), & R = L_{(1/2)}, & r^p = i^p_{(1/2)}, & \varphi_x = \varphi_x^{(1/2)} \text{ for } x \in R.
\end{array}
$$

Recall that by definition $M_* = L^1(M)_{\mathrm{left}}$ and $M = L^\infty(M)_{\mathrm{left}}$. From now on we consider M_* and M as subspaces of R^* by means of the respective maps r^*_∞ and r^*_1 and it is convenient to omit these maps in the notation. So the identifications of M_* and M in R^* are given by the pairings:

$$
\langle \omega, y\rangle_{R^*,R} = \quad \omega(y), \qquad \omega \in M_*, y \in R, \tag{5.8}
$$

$$
\langle x, y\rangle_{R^*,R} = \quad \varphi_y(x), \qquad x \in M, y \in R. \tag{5.9}
$$

The norm on L will be denoted by $\| \cdot \|_L$.

In Section 5.8 we need to consider M_* and M as subspaces of L^* by means of the respective maps l^*_∞ and l^*_1. Analogous to the previous discussion, we will regard M_* and M as subspaces of L^* by means of the pairings:

$$
\langle \omega, y\rangle_{L^*,L} = \quad \omega(y), \qquad \omega \in M_*, y \in L, \tag{5.10}
$$

$$
\langle x, y\rangle_{L^*,L} = \quad {}_y\varphi(x), \qquad x \in M, y \in L. \tag{5.11}
$$

5.4 Comparison with Hilsum's L^p-spaces

Here, we recall the definition of non-commutative L^p-spaces given in [39], see also [78]. We need these spaces for two reasons.

First of all, many of the objects we introduce are constructed by means of Theorem 5.1.5. For that reason the structures are abstract in nature. The advantage of the Hilsum approach is that it is much more concrete. Hence, also the objects defined in Section 5.6 have a more concrete meaning when they are considered in the Hilsum setting.

Secondly, a non-commutative L^2-space associated with a von Neumann algebra M with weight φ can be identified with the GNS-space $\mathcal{H}$ of the weight. In [78, Theorem 23] this identification is given for Hilsum's definition. Izumi [40] does not explicitly keep track of an isomorphism between $L^2(M)_{\text{left}}$ with $\mathcal{H}$. Here we make this isomorphism explicit. This is useful for the L^p-Fourier transform. In particular, Corollary 5.6.6, of which the proof appears to be surprisingly easy, relies heavily on this identification.

We refer to the original paper [39] for Hilsum's L^p-spaces. The following is also nicely summarized in [77, Sections III and IV]. Fix a normal semi-finite faithful weight ϕ on the commutant M'. Let σ^ϕ be its modular automorphism group.

Definition 5.4.1. Consider a closed, densely defined operator x on $\mathcal{H}$. x is called γ-homogeneous, with $\gamma \in \mathbb{R}$ if the following skew commutation relation holds

$$ax \subseteq x\sigma^{\phi}_{i\gamma}(a), \qquad \text{for all } a \in M' \text{ analytic w.r.t. } \sigma^\phi.$$

The following theorem requires the spatial derivate [75]. The definition is recalled in Appendix A.5. For a good and more elaborate introduction (without proofs however) we refer to [78, Section III]. The spatial derivative construction gives a passage between M_* and the (-1)-homogeneous operators. The following theorem can be found under the given references in [77]. It can be derived from [15, Theorem 13].

Theorem 5.4.2 (Theorem 29, Definition 33 and Corollary 34 of [77])**.** *Let x be a closed densely defined operator with polar decomposition $x = u|x|$. Let $p \in [1, \infty]$. The following are equivalent:*

1. *x is $(-1/p)$-homogeneous;*
2. *$u \in M$ and $|x|^p$ is (-1)-homogeneous;*
3. *$u \in M$ and there is a normal semi-finite weight ψ on M such that $|x|^p$ equals the spatial derivative $d\psi/d\phi$.*

Definition 5.4.3. Let $p \in [1, \infty)$. The Hilsum L^p-space $L^p(\phi)$ is defined as the space of closed densely defined operators x on the GNS-space $\mathcal{H}$ of φ such that if $x = u|x|$ is the polar decomposition, then $|x|^p$ is the spatial derivative of a positive $\omega \in M_*$ and $u \in M$. It carries the norm $\|x\|_p = (\omega(1))^{1/p}$. We set $L^\infty(M) = M$.

In particular, every operator in $L^p(\phi)$ is closed densely defined and $(-1/p)$-homogeneous. This includes $p = \infty$. By Theorem 5.4.2, the spatial derivative gives an isometric isomorphism between M_* and $L^1(\phi)$.

If x is a positive self-adjoint operator that is -1-homogeneous, we define an integral by

$$\int x \, d\phi = \psi(1),$$

where ψ is the unique normal semi-finite weight on M such that $x = d\psi/d\phi$. We extend the integral linearly to $L^1(\phi)$, where it takes finite values.

Theorem 5.4.4 (Theorem IV.15 of [78]). *Let $p \in [1,\infty)$ and set $q \in [1,\infty)$ by $1/p + 1/q = 1$. Suppose that $x \in L^p(\phi)$ and $y \in L^q(\phi)$, then $x \cdot y \in L^1(\phi)$. Moreover,*

$$\langle x, y\rangle_{L^p(\phi),L^q(\phi)} = \int x \cdot y d\phi,$$

identifies $L^q(\phi)$ as the dual Banach space of $L^p(\phi)$.

We introduce notation for the distinguished spatial derivative

$$d = d\varphi/d\phi.$$

d is a strictly positive self-adjoint operator acting on the GNS-space $\mathcal{H}$. We need the fact that it implements the modular automorphism group of φ and ϕ, i.e.

$$\sigma_t(x) = d^{it} x d^{-it}, \ x \in M \qquad \sigma_t^\phi(x) = d^{-it} y d^{it}, \ y \in M'.$$

Using this, one can prove that d is (-1)-homogeneous, see [78, Lemma 22]. We emphasize that d is only in $L^1(\phi)$ if φ is a state. The operator d forms a handy tool to find elements of $L^p(\phi)$.

Lemma 5.4.5 (Theorem 26 of [78]). *Let $p \in [2,\infty]$ and let $x \in \mathfrak{n}_\varphi$. Then, $xd^{1/p}$ is preclosed and its closure $[xd^{1/p}]$ is in $L^p(\phi)$. Moreover, there is an isometric isomorphism $\mathcal{P} : \mathcal{H} \to L^2(\phi)$ given by $[xd^{1/2}] \mapsto \Lambda(x)$.*

We use this result to prove the following.

Proposition 5.4.6. *Let $p \in [1,\infty]$.*

1. *Let $a, b \in \mathcal{T}_\varphi$. Then $abd^{1/p}$ is preclosed and its closure $[abd^{1/p}]$ is in $L^p(\phi)$.*
2. *Let $a, b \in \mathcal{T}_\varphi$ and let $x \in \mathfrak{n}_\varphi$. Let $p \in [1,2]$ and set $q \in [2,\infty)$ by the equality $1/p + 1/q = 1$. We can compute the following pairing explicitly:*

$$\langle [xd^{1/q}], d^{1/p} ab\rangle_{L^q(\phi),L^p(\phi)} = \int [xd^{1/q}] \cdot d^{1/p} ab \, d\phi = \varphi(abx).$$

3. *There is an isometric isomorphism $\Phi_p : L^p(\phi) \to L^p(M)_{\mathrm{left}}$ such that*

$$\Phi_p : [abd^{1/p}] \mapsto l^p(ab), \qquad a, b \in \mathcal{T}_\varphi.$$

4. *There is an isometric isomorphism $\Psi_p : L^p(\phi) \to L^p(M)_{\mathrm{right}}$ such that*

$$\Psi_p : [d^{1/p}ab] \mapsto r^p(ab), \qquad a, b \in \mathcal{T}_\varphi.$$

5. *There is a unitary map $U_l : L^2(M)_{\mathrm{left}} \to \mathcal{H}$ determined by*

$$U_l : l^2(a) \mapsto \Lambda(a), \qquad a \in \mathcal{T}_\varphi^2.$$

6. *More general, there is a unitary map $U_{(z)} : L^2_{(z)}(M) \to \mathcal{H}$ determined by*

$$U_{(z)} : i^2_{(z)}(a) \mapsto \Lambda(\sigma_{-i(z/2+1/4)}(a)), \qquad a \in \mathcal{T}_\varphi^2.$$

7. *More particular, there is a unitary map $U_r : L^2(M)_{\mathrm{right}} \to \overline{\mathcal{H}}$, $\overline{\mathcal{H}}$ being the conjugate Hilbert space of $\mathcal{H}$, determined by*

$$U_r : r^2(a) \mapsto \overline{\Lambda(a^*)}, \qquad a \in \mathcal{T}_\varphi^2.$$

Proof. (1) First note that using [78, Lemma 22] for the first inclusion, Lemma 5.4.5 for the third and [39, Theorem 4 (3)] for the last,

$$abd^{1/p} \subseteq d^{2/p}\sigma_{2i/p}(ab)d^{2/p} \subseteq d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}],$$

which is an element in $L^{p/2}(\phi) \cdot L^{p/2}(\phi) \subseteq L^p(\phi)$. Hence,

$$(abd^{1/p})^* \supseteq (d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}])^* \in L^p(\phi). \tag{5.12}$$

So that $(abd^{1/p})^*$ is densely defined. Hence $abd^{1/p}$ is preclosed and by (the proof of) [39, Theorem 4 (1)], $(abd^{1/p})^* = (d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}])^*$, hence $[abd^{1/p}] = d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}]$.

(2) First of all, note that $[xd^{1/q}] \cdot d^{1/p}ab \supseteq xd^{1/q}d^{1/p}ab = xdab$. Since $x \in L^\infty(\phi)$ and $dab \in L^1(\phi)$, we know that $xdab$ is closable and hence,

$$L^1(\phi) \ni [xd^{1/q}] \cdot d^{1/p}ab \supseteq x \cdot dab \in L^1(\phi).$$

Hence, by [39, Théorème 4] we see that $[xd^{1/q}] \cdot d^{1/p}ab = x \cdot dab$. Since p was choosen arbitrary, we find $[xd^{1/q}] \cdot d^{1/p}ab = [xd^{1/2}] \cdot d^{1/2}ab$. We use the trace property of the integral to find

$$\int [xd^{1/q}] \cdot d^{1/p}ab \, d\phi = \int [xd^{1/2}] \cdot d^{1/2}ab \, d\phi = \int d^{1/2}ab \cdot [xd^{1/2}] \, d\phi. \tag{5.13}$$

Next, it is explained on [78, p. 346] that for $x \in \mathfrak{n}_\varphi$, we have $\int d^{1/2}x^* \cdot [xd^{1/2}]\, d\phi = \varphi^{(0)}_{x^*x}(1)$, which in turn equals $\varphi(x^*x)$ by [78, Eqn. (6)]. By using a polarization argument, we find that for $x, y \in \mathfrak{n}_\varphi$ we have

$$\int d^{1/2}y^* \cdot [xd^{1/2}]\, d\phi = \varphi(y^*x). \tag{5.14}$$

Combining (5.13) with (5.14) yields the statement.

(3) It is argued in the remarks following [40, Proposition 2.4] that (5.2) for $z = 0$ equals the compatible couple as considered in [78]. First note that by [78, Eqn. (50)],

$$[abd^{1/p}] = d^{2/p}\sigma_{2i/p}(a) \cdot [\sigma_{2i/p}(b)d^{2/p}] = \mu_p(\sigma_{2i/p}(ab)),$$

where μ_p is the embedding of $L_{(0)}$ in $L^p(\phi)$, see [78, Theorem 27]. The main result of [78] is that $L^p(\phi)$ is isometrically isomorphic to $L^p_{(0)}(M)$. The isomorphism is given by the map $\nu_p : L^p(\phi) \to L^p_{(0)}(M)$ of [78, Theorem 30]. Moreover, we see that $\nu_p\mu_p = (i^1_{(0)})^* i^\infty_{(0)}$ by commutativity of [78, Eqn. (55)]. In turn we have $(i^1_{(0)})^* i^\infty_{(0)} = i^p_{(0)}$ by commutativity of (5.2). Hence, we have an isometric isomorphism $L^p(\phi) \to L^p_{(0)}(M)$ for which

$$[abd^{1/p}] \mapsto \nu_p([abd^{1/p}]) = \nu_p\mu_p(\sigma_{2i/p}(ab)) = i^p_{(0)}(\sigma_{2i/p}(ab)).$$

We conclude the proof by applying the isometric isomorphism $U_{(-1/2,0)}$ of Theorem 5.2.7, so that we we get an isometric isomorphism

$$\Phi_p : L^p(\phi) \to L^p_{(-1/2)}(M) = L^p(M)_{\text{left}},$$

such that

$$\Phi_p : [abd^{1/p}] \mapsto U_{(-1/2,0)} i^p_{(0)}(\sigma_{2i/p}(ab)) = i^p_{(-1/2)}(ab) = l^p(ab), \qquad a, b \in \mathcal{T}_\varphi.$$

(4) Put $\Psi_p = U_{(1/2,-1/2)}\Phi_p$. The precise proof relies on the same technicalities as we have used earlier in this proof.

(5) This follows from (3) by applying Lemma 5.4.5 and the fact that $\Lambda(\mathcal{T}^2_\varphi)$ is dense in $\mathcal{H}$. So $U_l = \Phi_p^{-1}\mathcal{P}^{-1}$.

(6) $U_{(z)} = U_l U_{2,(-1/2,z)}$.

(7) U_r is the composition of $JU_{(z)}$ and the map $\mathcal{H} \to \overline{\mathcal{H}} : \xi \mapsto \overline{\xi}$. □

Recall that $L^2(M)_{\text{left}}$ is by definition a subspace of R^*. Therefore we can pair elements of $L^2(M)_{\text{left}}$ with elements of R. Similarly, elements of $L^2(M)_{\text{right}}$ can be paired with elements of L.

Proposition 5.4.7.

1. *For $\xi \in \mathcal{H}$, $y \in R$,*

$$\langle U_l^*\xi, y\rangle_{R^*,R} = \langle \xi, \Lambda(y^*)\rangle.$$

2. For $\overline{\xi}$ in the conjugate Hilbert space $\overline{\mathcal{H}}$ and $y \in L$,

$$\langle U_r^* \overline{\xi}, y \rangle_{L^*,L} = \langle \Lambda(y), \xi \rangle.$$

Proof. (1) First assume that $\xi = \Lambda(x) = U_l l_2(x), x \in L$. Using the commutativity of (5.2) in the second equality,

$$\begin{aligned} \langle U_l^* \xi, y \rangle_{R^*,R} &= \langle l_2(x), y \rangle_{R^*,R} = \langle l_1(x), y \rangle_{R^*,R} \\ &\overset{(5.8)}{=} ({}_x\varphi)(y) \overset{(5.7)}{=} \varphi(yx) = \langle \Lambda(x), \Lambda(y^*) \rangle = \langle \xi, \Lambda(y^*) \rangle. \end{aligned}$$

The proposition follows by the fact that $\Lambda(\mathcal{T}_\varphi^2) \subseteq \Lambda(L)$ is dense in $\mathcal{H}$.

(2) The proof is completely analogous. □

Notation 5.4.8. From now on, we identify $\mathcal{H}$ and $L^2(M)_{\mathrm{left}}$ and consider it as a subspace of R^*. The identification is given via the unitary U_l. Under this identification the map l_2 becomes the GNS-map Λ, see Proposition 5.4.6. By Proposition 5.4.7 we see that $\mathcal{H}$ is identified as a subspace of R^* by means of the pairing

$$\langle \xi, y \rangle_{R^*,R} = \langle \xi, \Lambda(y^*) \rangle \qquad \xi \in \mathcal{H}, y \in R. \tag{5.15}$$

Similarly, we identify $\overline{\mathcal{H}}$ with $L^2(M)_{\mathrm{right}}$ using U_r. $\overline{\mathcal{H}}$ is a subspace of L^* by means of the pairing

$$\langle \overline{\xi}, y \rangle_{L^*,L} = \langle \Lambda(y), \xi \rangle \qquad \overline{\xi} \in \overline{\mathcal{H}}, y \in L. \tag{5.16}$$

5.5 Intersections of L^p-spaces

As indicated in the introduction of the chapter, the intersections of the various L^p-spaces depend on the interpolation parameter z of Definition 5.2.1. Here we study the intersections of $L^1(M)_{\mathrm{left}}$ and $L^2(M)_{\mathrm{left}}$, as well as the intersections of $L^2(M)_{\mathrm{left}}$ and $L^\infty(M)_{\mathrm{left}}$. The spaces turn out to be natural and well-known in the theory of locally compact quantum groups. We use the intersections in order to apply the re-iteration theorem, see [3].

Notation 5.5.1. In this section, any interpolation space should be understood with respect to the diagram in (5.2) for the parameter $z = -1/2$. Recall that we introduced short hand notation for this diagram in Notations 5.3.3 and 5.4.8. Moreover, we *identified* $M_*, \mathcal{H}$ and M as subspaces of R^* by means of the pairings (5.8), (5.15) and (5.9). Similarly, any intersection of two such spaces should be understood as an intersection within R^*. The notation can be summarized by means of the non-dotted arrows in (5.17). The dotted part of the diagram is the main topic of the present chapter.

$$\begin{tikzcd}
& & M_* \arrow[ddr, bend left, "(5.8)"] & \\
& \mathcal{I} \arrow[ur, dashed] \arrow[r, dashed, "s^p"] \arrow[dr, dashed, "\xi"] & L^p(M)_{\mathrm{left}} \arrow[dr] & \\
L \arrow[uurr, bend left, "l^1"] \arrow[ur, dashed, "l^1"'] \arrow[urr, "l^p"] \arrow[rr, "\Lambda"] \arrow[drr, "l^q"'] \arrow[dr, dashed] \arrow[ddrr, bend right, "l^\infty"'] & & \mathcal{H} \arrow[r, "(5.15)"] & R^*; \\
& \mathfrak{n}_\varphi \arrow[ur, dashed, "\Lambda"] \arrow[r, dashed, "t^q"] \arrow[dr, dashed] & L^q(M)_{\mathrm{left}} \arrow[ur] & \\
& & M \arrow[uur, bend right, "(5.9)"'] &
\end{tikzcd} \tag{5.17}$$

The following set defines the intersection of M_* and $\mathcal{H}$. Note that the definition was already stated in Chapter 1. For convenience, we repeat it here.

Definition 5.5.2. We set:

$$\mathcal{I} = \{\omega \in M_* \mid \Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi, \text{ is bounded } \}.$$

By the Riesz theorem, for every $\omega \in \mathcal{I}$, there exists a $\xi(\omega) \in \mathcal{H}$ such that $\langle \xi(\omega), \Lambda(x) \rangle = \omega(x^*)$.

Theorem 5.5.3. *We have $\mathcal{I} = \mathcal{H} \cap M_*$, where the equality should be interpreted within R^*, see Notation 5.5.1. Within R^*, $\omega \in \mathcal{I}$ equals $\xi(\omega) \in \mathcal{H}$.*

Proof. We first prove $\supseteq$. Let $\xi \in \mathcal{H}$ and $\omega \in M_*$ be such that $\xi = \omega$ in R^*. For $y \in R$,

$$\omega(y) \overset{(5.8)}{=} \langle \omega, y \rangle_{R^*,R} = \langle \xi, y \rangle_{R^*,R} \overset{(5.15)}{=} \langle \xi, \Lambda(y^*) \rangle.$$

L contains $\mathcal{T}_\varphi^2$. Moreover, $\mathcal{T}_\varphi^2$ is a σ-strong-$*$/norm core for Λ, see Lemma A.6.3. Hence, it follows that $\omega \in \mathcal{I}$.

To prove $\subseteq$, let $\omega \in \mathcal{I}$. For $y \in R$,

$$\langle \xi(\omega), y \rangle_{R^*,R} \overset{(5.15)}{=} \langle \xi(\omega), \Lambda(y^*) \rangle = \omega(y) \overset{(5.8)}{=} \langle \omega, y \rangle_{R^*,R}.$$

Hence, $\xi(\omega) = \omega$ in R^*. □

Note that $(M_*, \mathcal{H})$ forms a compatible couple. As explained in Section 5.1, the intersection of these two spaces carries a natural norm for which it is a Banach space. So, for $\omega \in \mathcal{I}$ we define

$$\|\omega\|_{\mathcal{I}} = \max\{\|\omega\|, \|\xi(\omega)\|\}.$$

Proposition 5.5.4. *The map $k : L \to \mathcal{I} : x \mapsto {}_x\varphi$ is injective, norm-decreasing and has dense range. In fact, $k(\mathcal{T}_\varphi^2)$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}$.*

Proof. Suppose that $x \in L$ and ${}_x\varphi = 0$, then $0 = ({}_x\varphi)(x^*) = \varphi(x^*x)$. So $x = 0$ and hence k is injective. For $x \in L$, $\|{}_x\varphi\| \leq \|x\|_L$ and

$$\|\xi({}_x\varphi)\| = \|\Lambda(x)\| = \|{}_x\varphi(x^*)\|^{1/2} \leq \|{}_x\varphi\|^{1/2}\|x^*\|^{1/2} \leq \|x\|_L,$$

so that k is norm-decreasing. Now we prove that the range of k is dense in $\mathcal{I}$. We identify $\mathcal{I}$ with the subspace $\{(\omega, \xi(\omega)) \mid \omega \in \mathcal{I}\} \subseteq M_* \times \mathcal{H}$. We equip $M_* \times \mathcal{H}$ with the norm $\|(\omega,\xi)\|_{\max} = \max\{\|\omega\|, \|\xi\|\}$. The norm coincides with $\|\cdot\|_{\mathcal{I}}$ on $\mathcal{I}$. The dual of $(M_* \times \mathcal{H}, \|\cdot\|_{\max})$ can be identified with $(M \times \mathcal{H}^*, \|\cdot\|_{\text{sum}})$, where $\|(x,\xi)\|_{\text{sum}} = \|x\| + \|\xi\|$. Let $N \subseteq M \times \mathcal{H}^*$ be the space of all (y,η) such that $\langle(\omega,\xi(\omega)),(y,\eta)\rangle_{M_*\times\mathcal{H}, M\times\mathcal{H}^*} = 0$ for all $\omega \in \mathcal{I}$. The dual of $\mathcal{I}$ is given by $(M \times \mathcal{H})/N$ equipped with the quotient norm.

Now, let $(y,\eta) \in M \times \mathcal{H}$ be such that

$$\langle({}_x\varphi, \Lambda(x)),(y,\eta)\rangle_{M_*\times\mathcal{H}, M\times\mathcal{H}^*} = ({}_x\varphi)(y) + \langle\Lambda(x),\eta\rangle = 0$$

for all $x \in \mathcal{T}_\varphi^2$. The proof is finished if we can show that $(y,\eta) \in N$. In order to do this, let $(e_j)_{j\in J}$ be a net as in Lemma A.6.2. Put $a_j = \sigma_{-\frac{i}{2}}(e_j)$. By the assumptions on (y,η), for $x \in \mathcal{T}_\varphi$,

$$({}_{xa_j}\varphi)(y) = -\langle\Lambda(xa_j),\eta\rangle. \tag{5.18}$$

For the left hand side we find by Corollary 5.3.2,

$$({}_{xa_j}\varphi)(y) = \varphi(\sigma_i(a_j)yx) = \langle\Lambda(x), \Lambda(y^*\sigma_i(a_j)^*)\rangle, \tag{5.19}$$

where the first equality follows from [40, Proposition 2.3]. For the right hand side of (5.18) we find

$$\langle\Lambda(xa_j),\eta\rangle = \langle J\sigma_{-\frac{i}{2}}(a_j^*)J\Lambda(x),\eta\rangle = \langle\Lambda(x), J\sigma_{\frac{i}{2}}(a_j)J\eta\rangle. \tag{5.20}$$

Hence (5.18) together with (5.19) and (5.20) yield

$$\Lambda(y^*\sigma_i(a_j)^*) = -J\sigma_{\frac{i}{2}}(a_j)J\eta.$$

Hence, since $\sigma_{\frac{i}{2}}(a_j) = e_j \to 1$ strongly, $\Lambda(y^*\sigma_i(a_j)^*) \to -\eta$ weakly. For $\omega \in \mathcal{I}$,

$$\begin{aligned}\langle\xi(\omega),\eta\rangle &= -\lim_{j\in J}\langle\xi(\omega), \Lambda(y^*\sigma_i(a_j)^*)\rangle \\ &= -\lim_{j\in J}\omega(\sigma_i(a_j)y) = -\lim_{j\in J}\omega(\sigma_{\frac{i}{2}}(e_j)y) = -\omega(y).\end{aligned}$$

Thus $(y,\eta) \in N$. □

We turn to the intersection of $\mathcal{H}$ and M. It turns out that $\mathfrak{n}_\varphi$ is the intersection of M and $\mathcal{H}$.

Theorem 5.5.5. *We have $\mathfrak{n}_\varphi = \mathcal{H} \cap M$, where the equality should be interpreted within R^*, see Notation 5.5.1. Within R^*, $x \in \mathfrak{n}_\varphi$ equals $\Lambda(x) \in \mathcal{H}$.*

Moreover, let $\overline{L}$ be the closure of $l^\infty(L)$ in M. Then $\mathfrak{n}_\varphi = \mathcal{H} \cap \overline{L}$.

Proof. First we prove that $\mathfrak{n}_\varphi = \mathcal{H} \cap M$ in R^*. For $x \in \mathfrak{n}_\varphi, y \in R$,

$$\langle \Lambda(x), y\rangle_{R^*,R} \overset{(5.15)}{=} \langle \Lambda(x), \Lambda(y^*)\rangle = \varphi(yx) \overset{(5.7)}{=} \varphi_y(x) \overset{(5.9)}{=} \langle x, y\rangle_{R^*,R},$$

so $\Lambda(x) = x$ in R^*. Hence the inclusion $\subseteq$ follows.

Now let $x \in M$, $\xi \in \mathcal{H}$ be such that $x = \xi$ in R^*. For $y \in R$,

$$\varphi_y(x) \overset{(5.9)}{=} \langle x, y\rangle_{R^*,R} = \langle \xi, y\rangle_{R^*,R} \overset{(5.15)}{=} \langle \xi, \Lambda(y^*)\rangle.$$

For $a \in \mathcal{T}_\varphi^2$, $y \in \mathfrak{n}_\varphi$, using Corollary 5.3.2 for the third, fourth and fifth equality,

$$\begin{aligned}\langle \Lambda(xa), \Lambda(y)\rangle &= \varphi(y^*xa) = {}_a\varphi(y^*x) = \varphi_{\sigma_i(a)}(y^*x)\\ =&\varphi_{\sigma_i(a)y^*}(x) = \langle \xi, \Lambda(y\sigma_i(a)^*)\rangle = \langle J\sigma_{i/2}(a)^*J\xi, \Lambda(y)\rangle.\end{aligned}$$

So for $a \in \mathcal{T}_\varphi^2$, $\Lambda(xa) = J\sigma_{i/2}(a)^*J\xi$. Let $(e_j)_{j\in J}$ be a net as in Lemma A.6.2. Put $a_j = e_j^2$. Then $xa_j \to x$ σ-weakly. Furthermore, $J\pi(\sigma_{i/2}(a_j)^*)J\xi \to \xi$ weakly, hence $\Lambda(xa_j)$ converges weakly. Since Λ is σ-weak/weak closed, $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$ and $\xi = \Lambda(x)$. This proves $\supseteq$.

Recall the complex interpolation method from Definition 5.1.4. Recall that in this section every interpolation space should be interpreted with respect to (5.2) for parameter $z = -1/2$. Note that [3, Theorem 4.2.2] gives the second equality in

$$\mathcal{H} = (M, M_*)_{[1/2]} = (\overline{L}, M_*)_{[1/2]} \subseteq \overline{L} + M_*. \tag{5.21}$$

We now prove that

$$M \cap (\overline{L} + M_*) = \overline{L}. \tag{5.22}$$

Take any $s \in M \cap (\overline{L} + M_*) \subseteq R^*$. Since $s \in \overline{L} + M_*$, we can choose representatives $x \in \overline{L}, \omega \in M_*$ such that $s = x + \omega$ in R^*. Since $s \in M$, we can find a representative $y \in M$ such that $s = y$ in R^*. Then $\omega = y - x$ is both in M_* and M, and hence by (5.4) in $M_* \cap M = L$. Hence we see that $s = x + \omega \in \overline{L} + L = \overline{L}$. This proves $\subseteq$, the other inclusion is trivial.

Now, (5.21) and (5.22) imply:

$$\mathcal{H} \cap M = \mathcal{H} \cap M \cap (\overline{L} + M_*) = \mathcal{H} \cap \overline{L}.$$

□

Again, we introduce the norm on an intersection of a compatible couple as in Section 5.1. For $x \in \mathfrak{n}_\varphi$, we put

$$\|x\|_{\mathfrak{n}_\varphi} = \max\{\|x\|, \|\Lambda(x)\|\}.$$

Again we can prove a density result similar to Propostion 5.5.4. The proof is subtle.

Proposition 5.5.6. *The map $k' : L \to \mathfrak{n}_\varphi : x \mapsto x$ is injective, norm-decreasing and has dense range.*

Proof. The non-trivial part is that $k'(L)$ is dense in $\mathfrak{n}_\varphi$ with respect to $\|\cdot\|_{\mathfrak{n}_\varphi}$. To prove this, we identify $\mathfrak{n}_\varphi$ with the subspace $\{(x, \Lambda(x)) \mid x \in \mathfrak{n}_\varphi\} \subseteq M \times \mathcal{H}$. For $(x, \xi) \in M \times \mathcal{H}$, we set $\|(x,\xi)\|_{\max} = \max\{\|x\|, \|\xi\|\}$. So $\|\cdot\|_{\max}$ coincides with $\|\cdot\|_{\mathfrak{n}_\varphi}$ on $\mathfrak{n}_\varphi$. The dual of $(M \times \mathcal{H}, \|\cdot\|_{\max})$ is given by $(M^* \times \mathcal{H}^*, \|\cdot\|_{\mathrm{sum}})$, where $\|(\theta, \xi)\|_{\mathrm{sum}} = \|\theta\| + \|\xi\|$.

Let $(\theta, \xi) \in M^* \times \mathcal{H}^*$ be such that for all $x \in L$,

$$\theta(x) + \langle \Lambda(x), \xi \rangle = 0. \tag{5.23}$$

We must prove that (5.23) holds for all $x \in \mathfrak{n}_\varphi$. The proof proceeds in several steps.

CLAIM I: There exists an $\omega \in M_*$ such that for $x \in L \cap R$, $\omega(x) = \theta(x)$.
Proof of the claim. From Corollary 5.3.2 it follows that

$$\overline{L \cap R} \left(= \overline{l^\infty(L) \cap r^\infty(R)} \right)$$

is a C*-algebra. Here and in the rest of this proof the closure has to be interpreted within M. Let $(u_j)_{j \in J}$ be an approximate unit for the C*-algebra $\overline{L \cap R}$. We may assume that $u_j \in (L \cap R)^+$. Set $\omega_j(x) = -\langle x\Lambda(u_j), \xi \rangle, x \in M$. So $\omega_j \in M_*$. Moreover, by (5.23) and Corollary 5.3.2,

$$\omega_j(x) = -\langle x\Lambda(u_j), \xi \rangle = \theta(xu_j).$$

Let ρ be a representation of $\overline{L \cap R}$ on a Hilbert space $\mathcal{H}_\rho$ such that $\theta(x) = \langle \rho(x)\xi, \eta \rangle$ for certain vectors $\xi, \eta \in \mathcal{H}_\rho$. Then $\omega_j(x) = \langle \rho(x)\rho(u_j)\xi, \eta \rangle$. Since $\rho(u_j) \to 1$ strongly, $\|\omega_j|_{\overline{L \cap R}} - \theta|_{\overline{L \cap R}}\| \to 0$. $L \cap R (\supseteq \mathcal{T}_\varphi^2)$ is σ-weakly, hence strongly dense in M so that by Kaplansky's density theorem $\|\omega_j\| = \|\omega_j|_{L \cap R}\|$. Hence $(\omega_j)_{j \in J}$ is a Cauchy net in M_*. Let $\omega \in M_*$ be its limit. This proves the first the claim.

CLAIM II: For $x \in \mathfrak{n}_\varphi$, we find $\omega(x) = -\langle \Lambda(x), \xi \rangle$.
Proof of the claim. Note that $L \cap R$ is a σ-weak/weak core for Λ. Indeed, $\mathcal{T}_\varphi^2$ is contained in $L \cap R$ so that we can apply Lemma A.6.3.

Now, if $x \in L \cap R$, the claim follows by the first claim and the properties of θ, i.e. $\omega(x) = \theta(x) = -\langle \Lambda(x), \xi \rangle$. Let $x \in \mathfrak{n}_\varphi$ and, by the previous paragraph, let $(x_i)_{i \in I}$ be a net in $L \cap R$ converging σ-weakly to x such that $\Lambda(x_i) \to \Lambda(x)$ weakly. Then, we arrive at the following equation:

$$\omega(x) = \lim_{i \in I} \omega(x_i) = -\lim_{i \in I} \langle \Lambda(x_i), \xi \rangle = -\langle \Lambda(x), \xi \rangle. \tag{5.24}$$

This proves the second claim.

CLAIM III: $\overline{L \cap R} = \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} = \overline{L} \cap \overline{R}$, where the closures are interpreted with respect to the norm on M.
Proof of the claim. Note that by Proposition 5.3.1, $\overline{L \cap R} \subseteq \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*}$. By Theorem 5.5.5, we see that $\mathfrak{n}_\varphi \subseteq \overline{L}$. Since $R = \{x^* \mid x \in L\}$, see Corollary 5.3.2, we also have $\mathfrak{n}_\varphi^* \subseteq \overline{R}$. Hence,

$$\overline{L \cap R} \subseteq \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} \subseteq \overline{L} \cap \overline{R} \qquad \text{(closures in } M).$$

The inclusions are in fact equalities. Indeed, let $x \in \overline{L} \cap \overline{R}$ be positive. Let x_n and y_n be sequences in L, respectively R, converging in norm to x. Then, by Corollary 5.3.2, $y_n x_n \in LR \subseteq L \cap R$. $y_n x_n$ is norm convergent to x^2. So $x^2 \in \overline{L \cap R}$, hence $x \in \overline{L \cap R}$. From Corollary 5.3.2 it follows that $\overline{L \cap R}$ and $\overline{L} \cap \overline{R}$ are C*-algebras. Hence, $\overline{L \cap R} = \overline{\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*} = \overline{L} \cap \overline{R}$.

CLAIM IV: Equation (5.23) holds for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.
Proof of the claim. Let $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and by the third claim, let $x_n \in L \cap R$ be a sequence converging in norm to x. Then, using the first claim in the second equality and the third claim in the fourth equality,

$$\theta(x) = \lim_{n\to\infty} \theta(x_n) = \lim_{n\to\infty} \omega(x_n) = \omega(x) = -\langle \Lambda(x), \xi \rangle.$$

Hence (5.23) follows for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

Remainder of the proof of the proposition. Let $x \in \mathfrak{n}_\varphi$ and let $x = u|x|$ be its polar decomposition. Since (5.23) holds for $y \in L$, we find for $y \in L$ that $uy \in L$ by Corollary 5.3.2 and,

$$(\theta u)(y) + \langle \Lambda(y), u^*\xi \rangle = 0, \tag{5.25}$$

where we defined $\theta u \in M$ by $(\theta u)(a) = \theta(ua), a \in M$. If we apply claim four to the pair $(\theta u, u^*\xi)$, we see that actually (5.25) holds for all $y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. In particular, putting $y = |x|$, the required equation (5.23) follows. □

Next, we apply the re-iteration theorem, see [3, Theorems 4.6.1], for the complex interpolation method to obtain $L^p(M)_{\mathrm{left}}, p \in (1,2]$ as an interpolation space of $\mathcal{H}$ and M_*. Similarly, $L^p(M)_{\mathrm{left}}, p \in [2,\infty)$ as an interpolation space of $\mathcal{H}$ and M. Recall that in this section every intersection and interpolation is understood with respect to (5.2) for the parameter $z = -1/2$.

Theorem 5.5.7. *We have the following interpolation properties:*

1. *For $p \in (1,2]$, $(\mathcal{H}, M_*)_{[\frac{2}{p}-1]} = L^p(M)_{\mathrm{left}}$.*
2. *For $q \in [2,\infty)$, $(\mathcal{H}, M)_{[1-\frac{2}{q}]} = (M, \mathcal{H})_{[\frac{2}{q}]} = L^q(M)_{\mathrm{left}}$.*

Proof. (1) Recall that $\overline{L} = \overline{l^\infty(L)}$ denotes the closure of L in M. Recall from (5.4) that $M_* \cap M = L$. By [3, Theorem 4.2.2 (b)] we get the first equality of:

$$(\overline{L}, M_*)_{[\frac{1}{2}]} = (M, M_*)_{[\frac{1}{2}]} = L^2(M)_{\mathrm{left}} \simeq \mathcal{H}. \tag{5.26}$$

The latter isomorphism is the identification in Notation 5.4.8. On the other hand, we find:

$$(\overline{L}, M_*)_{[1]} = (M, M_*)_{[1]} \tag{5.27}$$

Since $l^1(L)$ is dense in M_*, see [40, Proposition 2.4], we find that (5.27) in turn equals M_* by [3, Proposition 4.2.2].

Note that the following three density assumptions are satisfied:

(i) $l^1(L)$ is dense in M_*, see [40, Proposition 2.4].

(ii) $l^\infty(L)$ is dense in $\overline{L}$ (trivial).

(iii) $l^1(L)$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}$ by Proposition 5.5.4. Moreover $\mathcal{I}$ is the intersection of M_* and $\mathcal{H}$ in R^*, see Theorem 5.5.3.

Hence, we have checked the assumptions of the re-iteration theorem [3, Theorems 4.6.1] which is used in the third equality,

$$\begin{aligned} L^p(M)_{\mathrm{left}} &= (M, M_*)_{[\frac{1}{p}]} = (\overline{L}, M_*)_{[\frac{1}{p}]} \\ &= ((\overline{L}, M_*)_{[\frac{1}{2}]}, (\overline{L}, M_*)_{[1]})_{[\frac{2}{p}-1]} = (\mathcal{H}, M_*)_{[\frac{2}{p}-1]}, \end{aligned}$$

(here the second equality follows again by [3, Theorem 4.2.2]).

(2) Completely analogously, using Theorem 5.5.5 and Proposition 5.5.6, one proves that

$$L^q(M)_{\mathrm{left}} = (\mathcal{H}, M)_{[1-\frac{2}{q}]},$$

which in turn equals $(M, \mathcal{H})_{[\frac{2}{q}]}$ by [3, Theorem 4.2.1]. □

We end this section by defining the embedding of $\mathcal{I}$ in $L^p(M)_{\mathrm{left}}$ for $p \in [1,2]$. Similarly, we define the embedding of $\mathfrak{n}_\varphi$ in $L^q(M)_{\mathrm{left}}$, $q \in [2, \infty)$.

Definition 5.5.8. Let $p \in [1,2]$ and let $q \in [2,\infty)$. Consider the maps:

$$(l^1(L) \subseteq \mathcal{I}) \to L^p(M)_{\mathrm{left}} : \quad l^1(x) \mapsto l^p(x), \qquad x \in L, \tag{5.28}$$

$$(l^\infty(L) \subseteq \mathfrak{n}_\varphi) \to L^q(M)_{\mathrm{left}} : \quad l^\infty(x) \mapsto l^q(x), \qquad x \in L. \tag{5.29}$$

Note that by Proposition 5.5.4, $l^1(L)$ is dense in $\mathcal{I}$ with respect to the $\|\cdot\|_{\mathcal{I}}$-norm. Furthermore, by Theorem 5.5.7, $L^p(M)_{\mathrm{left}}$ is a complex interpolation space for the diagram

$$\begin{tikzcd} & M_* \arrow[dr, hook] & \\ \mathcal{I} \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & L^p(M)_{\mathrm{left}} \arrow[r, hook] & R^*; \\ & \mathcal{H} \arrow[ur, hook] & \end{tikzcd}$$

It follows from the definition of the interpolation norm that (5.28) is norm decreasing, where we take $\| \cdot \|_{\mathcal{I}}$ for the norm on $\mathcal{I}$. We denote the continuous extension of (5.28) by $s^p : \mathcal{I} \to L^p(M)_{\text{left}}$. Note that by Theorem 5.5.3, $s^2(\omega) = \xi(\omega)$, $\omega \in \mathcal{I}$. Similarly, (5.29) extends to a continuous map $t^q : \mathfrak{n}_\varphi \to L^q(M)_{\text{left}}$.

5.6 Fourier theory

In this Section we define an L^p-Fourier transform. Our strategy is similar to the one defining the classical L^p-Fourier transform on locally compact abelian groups. We first define a L^1- and L^2-Fourier transform and show that they form a compatible pair of morphisms, see Remark 5.1.3. Then we apply the complex interpolation method to get a bounded L^p-Fourier transform for $p \in [1, 2]$,

$$\mathcal{F}_p : L^p(M)_{\text{left}} \to L^q(\hat{M})_{\text{left}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

The crucial property of the complex interpolation method is the one given by Theorem 5.1.5. The theorem gives the non-commutative analogue of the Riesz-Thorin theorem as mentioned in the introduction. It is for this reason that we have approached L^p-spaces from the perspective of interpolation spaces and that we have used Izumi's definition.

Notation 5.6.1. From now on, let (M, Δ) be a locally compact quantum group with left Haar weight φ. $(\hat{M}, \hat{\Delta})$ is the Pontrjagin dual. In this section all L^p-spaces we encounter are 'left' L^p-spaces which are defined with respect to the (dual) left Haar weight. More precisely, we stick to Notation 5.5.1. We equip the objects introduced in Sections 5.2 - 5.4 with a hat if they are associated with the dual quantum group. So we get $\hat{L}, \hat{R}, L^p(\hat{M})_{\text{left}}, \ldots$ Recall that by construction $\hat{\mathcal{H}} = \mathcal{H}$.

Theorem 5.6.2 (L^1- and L^2-Fourier transform). *We can define compatible Fourier transforms in the following way:*

1. *There exists a unique unitary map $\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$, which is determined by:*

$$\Lambda(x) \mapsto \hat{\Lambda}(\lambda({}_x\varphi)), \qquad x \in L. \tag{5.30}$$

2. *There exists a bounded map $\mathcal{F}_1 : M_* \to \hat{M} : \omega \mapsto \lambda(\omega)$. Moreover, $\|\mathcal{F}_1\| = 1$.*

3. *$\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$ and $\mathcal{F}_1 : M_* \to \hat{M}$ are compatible in the sense of Remark 5.1.3,*

i.e. the following diagram commutes:

$$\begin{tikzcd}
 & M_* \arrow[rrr, Rightarrow, "\mathcal{F}_1"] \arrow[dr, "(5.8)"] & & & \hat{M} \arrow[dr, "(5.9)"] & \\
L \arrow[ur, "l^1"] \arrow[r, "\Lambda"] & \mathcal{H} \arrow[r, "(5.15)"] \arrow[rrr, Rightarrow, bend right, "\mathcal{F}_2"'] & R^* & \hat{L} \arrow[ur, "\hat{l}^\infty"] \arrow[r, "\hat{\Lambda}"] & \mathcal{H} \arrow[r, "(5.15)"] & \hat{R}^*.
\end{tikzcd} \tag{5.31}$$

Proof. (1) By Proposition 5.5.3, we see that for $x \in L$, we have ${}_x\varphi \in \mathcal{I}$. By definition of $\hat{\Lambda}$, we have $\hat{\Lambda}(\lambda({}_x\varphi)) = \xi({}_x\varphi)$. Since by (5.7),

$${}_x\varphi(y^*) = \varphi(y^*x) = \langle \Lambda(x), \Lambda(y) \rangle, \qquad y \in \mathfrak{n}_\varphi,$$

we see that by definition of $\xi({}_x\varphi)$, we have $\xi({}_x\varphi) = \Lambda(x)$. So (5.30) is the identity map. Since $\mathcal{T}_\varphi^2 \subseteq L$ and $\Lambda(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{H}$, see the much stronger result of Lemma A.6.3, this determines a map on $\mathcal{H}$.

(2) The norm bound follows, since

$$\|\lambda(\omega)\| = \|(\omega \otimes \iota)(W)\| \leq \|(\omega \otimes \iota)\|\|W\| \leq \|\omega\|.$$

(3) For $y \in \hat{R}, x \in L$, we find

$$\begin{aligned}
&\langle \mathcal{F}_2 \Lambda(x), y \rangle_{\hat{R}^*, \hat{R}} = \langle \hat{\Lambda}(\lambda({}_x\varphi)), y \rangle_{\hat{R}^*, \hat{R}} \overset{(5.15)}{=} \langle \hat{\Lambda}(\lambda({}_x\varphi)), \hat{\Lambda}(y^*) \rangle \\
=&\hat{\varphi}(y\lambda({}_x\varphi)) \overset{(5.7)}{=} \hat{\varphi}_y(\lambda({}_x\varphi)) \overset{(5.9)}{=} \langle \lambda({}_x\varphi), y \rangle_{\hat{R}^*, \hat{R}} = \langle \mathcal{F}_1({}_x\varphi), y \rangle_{\hat{R}^*, \hat{R}},
\end{aligned}$$

which proves the commutativity of the diagram. $\square$

Remark 5.6.3. Kahng [43] defines an operator algebraic Fourier transform and in principle the idea behind Theorem 5.6.2 can also be found here. However, [43, Definition 3] has to be given a more careful interpretation, since, if φ is not a state, the expression $(\varphi \otimes \iota)(W(a \otimes 1)), a \in \hat{\lambda}(\hat{\mathcal{I}})$, is in general undefined. In case φ is a state, our definition of $\mathcal{F}_2$ equals Kahng's by Remark 5.2.2.

We comment on the classical situation. As is shown in Example 1.2.4, the Fourier transform is implicitly used to define the dual quantum group of a classical abelian group. It is for this reason that the L^2-Fourier transform $\mathcal{F}_2$ trivializes on the level of GNS-spaces. We work this out in the next example.

Example 5.6.4. Let G be a locally compact abelian group. For $f \in L^1(G)$, let ω_f be the the normal functional on $L^\infty(G)$ given by $\omega_f(g) = \int_G f(x)g(x)d_lx$. Then,

$$\lambda((\omega_f)) = (\omega_f \otimes \iota)(W) = \int_G f(x)\lambda_x d_lx \in \mathcal{L}(G),$$

where $x \mapsto \lambda_x$ is the left-regular representation. On the other hand, using the direct integral decomposition $L^\infty(\hat{G}) = \int_{\hat{G}}^{\oplus} \mathbb{C} d\pi$, we find for (5.1),

$$\hat{f} = \int_{\hat{G}}^{\oplus} \int_G f(x)\pi(x) dx \, d\pi = \int_G f(x) \int_{\hat{G}}^{\oplus} \pi(x) d\pi \, dx.$$

The left regular representation $x \mapsto \lambda_x$ is unitarily equivalent to $\int_{\hat{G}}^{\oplus} \pi d\pi$, where the intertwiner is given by the (classical) L^2-Fourier transform. Since the dual quantum group associated to a classical group is given by conjugating $L^\infty(\hat{G})$ with the classical L^2-Fourier transform, see Example 1.2.4, the identification of the dual GNS-space $L^2(\hat{G})$ with the GNS-space $L^2(G) = \mathcal{H}$ is given by applying the classical (inverse) L^2-Fourier transform. Hence, using these identifactions, we see that the transform defined in (5.1) is the quantum group analogue of the transform of Theorem 5.6.2.

Note that it is due to the identifications $L^2(M)_{\mathrm{left}}$ with $\mathcal{H}$ and $L^2(\hat{M})_{\mathrm{left}}$ with $\hat{\mathcal{H}} = \mathcal{H}$ that the L^2-Fourier transform becomes the identity map. If we had not made these identifications the map would be less trivial. It is for this reason that we have choosen to write *unitary map* in the first statement of Theorem 5.6.2 instead of *identity map*.

Remark 5.6.5. Our transform coincides with the definition given in [92, Definition 1.3]. To comment on this, suppose that (M, Δ) is compact, i.e. φ is a state. Let $A \subseteq M$ be the Hopf algebra of the underlying algebraic quantum group. We mention that A is the Hopf algebra of matrix coefficients of irreducible, unitary corepresentations of M and refer to [79] for more explanation. Let $\hat{A}$ be its dual, which is the space of linear functionals on A of the form $\varphi(\,\cdot\, x)$, where $x \in A$, [92, Theorem 1.2]. Van Daele defines the transform by

$$A \to \hat{A} : x \mapsto \varphi(\,\cdot\, x), \qquad x \in A.$$

On the other hand, by Remark 5.2.2, the normal functional ${}_x\varphi$, with $x \in L$ is given by $x\varphi$, since we assumed that φ is a state. Here $x\varphi \in M_*$ is defined by $(x\varphi)(y) = \varphi(yx), y \in M$ (we use this notation to distinguish it from the algebraic map $\varphi(\,\cdot\, x)$). So the Fourier transform is defined by:

$$x \mapsto \lambda(x\varphi), \qquad x \in M.$$

Since in the transition of compact algebraic quantum groups to compact von Neumann algebraic quantum groups, the element $\varphi(\,\cdot\, x) \in \hat{A}$ corresponds to $\lambda(x\varphi) \in \hat{M}$, this shows the correspondence. Here, we refer to [79] for compact algebraic quantum groups and their relations to locally compact quantum groups.

By Pontrjagin duality, one also find dual Fourier transforms. At every point in Theorem 5.6.2 where one of the objects $L, R, M, \varphi, \Lambda, \lambda, \mathcal{F}_2, \mathcal{F}_1$ appears, one should replace the object by the same object equipped with a hat and vise versa. In that way, we get a dual L^2-Fourier transform $\hat{\mathcal{F}}_2 : \mathcal{H} \to \mathcal{H}$, determined by

$$\hat{\Lambda}(x) \mapsto \Lambda(\hat{\lambda}({}_x\hat{\varphi})), \qquad x \in \hat{L}.$$

Since this map is on the GNS-level given by the identity, we automatically find the following corollary.

Corollary 5.6.6. *We have* $\mathcal{F}_2^{-1} = \hat{\mathcal{F}}_2$.

Next we apply the complex interpolation method to define L^p-Fourier transforms.

Theorem 5.6.7. *Let* $p \in [1, 2]$ *and set* q *by* $1/p + 1/q = 1$. *There exists a unique bounded linear map* $\mathcal{F}_p : L^p(M)_{\mathrm{left}} \to L^q(\hat{M})_{\mathrm{left}}$ *such that* $\mathcal{F}_p$ *is compatible with* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *in the sense of Remark 5.1.3, i.e. the following diagram commutes:*

$$\begin{tikzcd}
 & M_* \arrow[rrr, Rightarrow, "\mathcal{F}_1"] \arrow[dr, "(5.8)"] & & & \hat{M} \arrow[dr, "(5.9)"] & \\
L \arrow[ur, "l^1"] \arrow[r, "l^p"] \arrow[dr, "\Lambda"'] & L^p(M)_{\mathrm{left}} \arrow[r] \arrow[rrr, Rightarrow, bend right, "\mathcal{F}_p"] & R^* & \hat{L} \arrow[ur, "\hat{l}^1"] \arrow[r, "\hat{l}^q"] \arrow[dr, "\hat{\Lambda}"'] & L^q(\hat{M})_{\mathrm{left}} \arrow[r] & \hat{R}^* \\
 & \mathcal{H} \arrow[ur, "(5.15)"'] \arrow[rrr, Rightarrow, "\mathcal{F}_2"] & & & \mathcal{H} \arrow[ur, "(5.15)"'] &
\end{tikzcd} \tag{5.32}$$

Moreover, $\|\mathcal{F}_p\| \leq 1$.

Proof. We can apply the complex interpolation method with parameter $\theta = 2/p - 1 = 1 - 2/q$ to the pairs $(\mathcal{H}, M_*)$ and $(\mathcal{H}, \hat{M})$ which are compatible couples as in (5.31). By Theorem 5.5.7 the corresponding interpolation spaces are respectively $L^p(M)_{\mathrm{left}}$ and $L^q(\hat{M})_{\mathrm{left}}$.

Since by Theorem 5.6.7, $\mathcal{F}_1 : M_* \to \hat{M}$ and $\mathcal{F}_2 : \mathcal{H} \to \mathcal{H}$ are compatible, with respect to diagram (5.31), we can use Remark 5.1.3 to obtain a map $\mathcal{F}_p : L^p(M)_{\mathrm{left}} \to L^q(\hat{M})_{\mathrm{left}}$ with the desired properties. $\square$

Remark 5.6.8. The inequality $\|\mathcal{F}_p\| \leq 1$ is also known as the *Hausdorff-Young* inequality. For unimodular groups, the inequality was proved by Kunze in [55]. In this case the Haar weight and the dual Haar weight are tracial, so that one can fall back on the theory of non-commutative integration with respect to a trace. A similar inequality was obtained for measured groupoids in [4].

We conclude this section by giving the Fourier transform explicitly in terms of Hilsum's L^p-spaces. The proof is not very difficult, except for the fact that we need another technicality about the correspondence between Hilsum's L^p-spaces and interplation spaces. We have postponed the technical part, since it requires the results of Section 5.5. We sketch the proof of the main theorem and see how it can be derived from this technical lemma.

Theorem 5.6.9. *Let $p \in [1,2]$ and set q by $1/p + 1/q = 1$. Fix a normal semi-finite faithful weight ϕ on M' and $\hat{\phi}$ on $\hat{M}'$. Set the corresponding spatial derivatives $d = d\varphi/d\phi$ and $\hat{d} = d\hat{\varphi}/d\hat{\phi}$. Then,*

$$\hat{\Phi}_q \mathcal{F}_p \Phi_p^{-1} : L^p(\phi) \to L^q(\hat{\phi}) : [ad^{1/p}] \mapsto [\lambda({}_a\varphi)\hat{d}^{1/q}], \qquad a \in \mathcal{T}_\varphi^2.$$

Proof. Note that in Theorem 5.6.9, we see that $[ad^{1/p}]$ is in $L^p(\phi)$ by Proposition 5.4.6. Moreover, since ${}_a\varphi \in \mathcal{I}$, we see that $\lambda({}_a\varphi) \in \mathfrak{n}_{\hat{\varphi}}$. Therefore, $[\lambda({}_a\varphi)\hat{d}^{1/q}]$ is in $L^q(\hat{\phi})$ by Lemma 5.4.5. The theorem follows by a careful analyis of (5.32) involving Proposition 5.4.6. The proof then relies on the following lemma applied to the von Neumann algebra of the dual quantum group. □

Lemma 5.6.10. *Let $q \in [2, \infty)$ and $x \in \mathfrak{n}_\varphi$. Recall that Φ_q was defined in Proposition 5.4.6 and t^q was defined in Remark 5.5.8. Then,*

$$\Phi_q([xd^{1/q}]) = t^q(x).$$

Proof. Note that in Proposition 5.4.6 we proved the lemma in case $x \in \mathcal{T}_\varphi^2$.

By Lemma A.6.3, let $(a_j)_{j\in J}$ be a bounded net in $\mathcal{T}_\varphi^2$ such that $a_j \to x$ σ-weakly and such that $\Lambda(a_j) \to \Lambda(x)$ weakly. Then, by [78, Theorem 26],

$$\| [(a_j - x)d^{1/q}] \|_q \leq \max\{ \|a_j - x\|, \|\Lambda(a_j - x)\| \}. \tag{5.33}$$

Set $p \in [1,2]$ by $1/p + 1/q = 1$. Let $b, c \in \mathcal{T}_\varphi$. Then, using Proposition 5.4.6,

$$\int [(a_j - x)d^{1/q}] \cdot d^{1/p} bc \, d\phi = \varphi(bc(a_j - x)) = \varphi(c(a_j - x)\sigma_{-i}(b)) \to 0. \tag{5.34}$$

By Proposition 5.5.4 we see that $k(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{I}$. In turn by Theorem 5.5.7 and Lemma 5.1.7 we know that $s^p(\mathcal{I})$ is dense in $L^p(M)_{\mathrm{left}}$. Hence, $l^p(\mathcal{T}_\varphi^2)$ is dense in $L^p(M)_{\mathrm{left}}$. And so $\Phi_p^{-1}(l^p(\mathcal{T}_\varphi^2)) = [\mathcal{T}_\varphi^2 d^{1/p}]$ is dense in $L^p(\phi)$. Let $b \in \mathcal{T}_\varphi^2$. Then, $[bd^{1/p}] \subseteq d^{1/p}\sigma_{i/p}(b)$ and by a similar argument as in (the proof of) [39, Theorem 4 (1)] we actually have equality of operators. We conclude that the set $d^{1/p}\mathcal{T}_\varphi^2$ is dense in $L^p(M)_{\mathrm{left}}$.

Since by (5.33), $[(a_j - x)d^{1/q}]$ is a bounded net in $L^q(\phi)$ we see from (5.34) that it is weakly convergent to 0.

So far we have proved that

$$\Phi_q([xd^{1/q}]) = \lim_{j \in J} \Phi_q[a_j d^{1/q}] = \lim_{j\in J} l^q(a_j),$$

where the limits are in the weak sense. We are finished if $l^q(a_j) \to t^q(x)$. To see this let $y \in R$. Then, using commutativity of (5.17) for the first equation,

$$\langle l^q(a_j) - t^q(x), y \rangle_{R^*,R} = \langle a_j - x, y\rangle_{R^*,R} = \varphi_y(a_j - x) \to 0.$$

Hence, the limit of the weakly convergent net $l^q(a_j)$ equals $t^q(x)$. □

Remark 5.6.11. If M is a semi-finite von Neumann algebra, then for every normal, semi-finite, faithful trace τ on M there exists a normal semi-finite faithful trace τ' on M' such that

$$\frac{d\tau}{d\tau'} = 1.$$

Indeed, using formula (A.2) it follows that choosing $\tau'(x) = \tau(J_\tau x^* J_\tau), x \in (M')^+$ gives such a pair of traces. Here J_τ is the modular conjugation of τ.

Consider the quantum group $(M, \Delta) = (L^\infty(G), \Delta_G)$ of a unimodular group G. The Haar weight φ is tracial. Furthermore, the unimodularity implies that also the dual weight given by the Plancherel weight $\hat{\varphi}$ on the group von Neumann algebra $\mathcal{L}(G)$ is tracial.

This shows that Fourier theory on a unimodular group can be understood without the use of spatial derivatives.

5.7 Convolution product

We define convolutions of elements in $L^1(M)_{\mathrm{left}} = M_*$ with elements in the space $L^p(M)_{\mathrm{left}}$. We prove that the Fourier transform transfers the convolution product into a product on the dual quantum group.

Notation 5.7.1. We keep the notation as in Section 5.6, c.f. Notation 5.6.1.

Note that, since Izumi's L^p-spaces are defined by means of complex interpolation, there is a priori no multiplication on these spaces. We extend the multiplication of M to the L^p-setting in the following proposition. This seems to be the most natural definition of a multiplication in the L^p-setting. The proof of the following proposition is completely similar to the one of Theorem 5.6.2 (3) and 5.6.7.

Proposition 5.7.2. *We extend the product of M to the L^p-setting.*

1. *Let $x \in M$. The maps*

$$m_x^\infty : M \to M : y \mapsto xy,$$
$$m_x^1 : M_* \to M_* : \omega \mapsto x\omega,$$

are compatible in the sense of Remark 5.1.3, i.e. the non-dotted arrows in the following diagram commute:

$$\begin{tikzcd}
 & M_* \arrow[rrr, Rightarrow, "m_x^1"] \arrow[dr, "(5.8)"] & & & M_* \arrow[dr, "(5.8)"] & \\
L \arrow[ur, "l^1"] \arrow[r, "l^p"] \arrow[dr, "l^\infty"'] & L^p(M)_{\mathrm{left}} \arrow[r] \arrow[rrr, Rightarrow, dashed, bend right, "m_x^p"] & R^* & L \arrow[ur, "l^1"] \arrow[r, "l^p"] \arrow[dr, "l^\infty"'] & L^p(M)_{\mathrm{left}} \arrow[r] & R^*. \\
 & M \arrow[ur, "(5.9)"'] \arrow[rrr, Rightarrow, "m_x^\infty"] & & & M \arrow[ur, "(5.9)"'] &
\end{tikzcd} \tag{5.35}$$

2. *Let $p \in (1, \infty)$. There is a unique bounded map $m_x^p : L^p(M)_{\mathrm{left}} \to L^p(M)_{\mathrm{left}}$ that is compatible with m_x^∞ and m_x^1, i.e. the dotted arrow in (5.35) makes the diagram commutative.*

Definition 5.7.3. Let $x \in M$ and let $y \in L^p(M)_{\mathrm{left}}$. We will write xy for $m_x^p(y)$.

For $\omega_1, \omega_2 \in M_*$, we define the convolution product,

$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta.$$

This product is well-known in the theory of locally compact quantum groups. We show that it is possible to extend it to the L^p-setting for $p \in [1, 2]$. Moreover, the convolution product is turned into the product of Definition 5.7.3 by the Fourier transfrom.

Theorem 5.7.4. *Let $p \in [1, 2]$ and set $q \in [2, \infty]$ by $1/p + 1/q = 1$.*

1. *Let $x \in L$ and let $\omega \in M_*$. Then,*

$$\omega * ({}_x\varphi) \in \mathcal{I} \qquad \text{and} \qquad \xi(\omega * ({}_x\varphi)) = \lambda(\omega)\Lambda(x).$$

2. *Let $\omega \in M_*$. For the sake of symmetry in the notation, denote $\omega *^2$ for the bounded operator $\lambda(\omega) : \mathcal{H} \to \mathcal{H}$. Define $\omega *^1 : M_* \to M_* : \theta \mapsto \omega * \theta$. Then $\omega *^1$ and $\omega *^2$ are compatible, i.e. the non-dotted arrows in following diagram commute:*

$$\begin{tikzcd}
& M_* \arrow[dr, "(5.8)"] \arrow[rrr, Rightarrow, "\omega *^1"] & & & M_* \arrow[dr, "(5.8)"] & \\
L \arrow[ur, "l^1"] \arrow[r, "l^p"] \arrow[dr, "\Lambda"'] & L^p(M)_{\mathrm{left}} \arrow[r] \arrow[rrr, Rightarrow, dashed, bend right, "\omega *^p"] & R^* & L \arrow[ur, "l^1"] \arrow[r, "l^p"] \arrow[dr, "\Lambda"'] & L^p(M)_{\mathrm{left}} \arrow[r] & R^*. \\
& \mathcal{H} \arrow[ur, "(5.15)"'] \arrow[rrr, Rightarrow, "\omega *^2"] & & & \mathcal{H} \arrow[ur, "(5.15)"'] &
\end{tikzcd} \tag{5.36}$$

3. *There is a unique bounded operator $\omega *^p : L^p(M)_{\mathrm{left}} \to L^p(M)_{\mathrm{left}}$ that is compatible with $\omega *^1$ and $\omega *^2$, i.e. (5.36) commutes.*

4. *For $\omega \in M_*$, $a \in L^p(M)_{\mathrm{left}}$,*

$$\mathcal{F}_1(\omega)\mathcal{F}_p(a) = \mathcal{F}_p(\omega *^p a),$$

where the left hand side uses Definition 5.7.3 for $L^q(\hat{M})_{\mathrm{left}}$.

Proof. Let $\theta \in \hat{\mathcal{I}}$ and put $y = \hat{\lambda}(\theta) = (\iota \otimes \theta)(W^*)$. Now (2) follows from

$$\begin{aligned}
&(\omega * ({}_x\varphi))(y^*) = (\omega \otimes ({}_x\varphi))\Delta((\iota \otimes \theta)(W^*)^*) = (\omega \otimes ({}_x\varphi) \otimes \overline{\theta})(W_{13}W_{23}) \\
=&\overline{\theta(((\omega \otimes \iota)(W)(({}_x\varphi) \otimes \iota)(W))^*)} = \langle \hat{\Lambda}\left((\omega \otimes \iota)(W)(({}_x\varphi) \otimes \iota)(W)\right), \hat{\xi}(\theta)\rangle \\
=&\langle (\omega \otimes \iota)(W)\xi({}_x\varphi), \Lambda(y)\rangle = \langle \lambda(\omega)\Lambda(x), \Lambda(y)\rangle,
\end{aligned}$$

and the fact that $\{\Lambda((\iota \otimes \theta)(W^*)) \mid \theta \in \hat{\mathcal{I}}\}$ is dense in $\mathcal{H}$.

The compatibility in (2) follows directly from (1) using Theorem 5.5.3. (3) follows by applying Theorem 5.5.7 to (2). (4) For $\omega_1, \omega_2 \in M_*$, note that

$$\begin{aligned} &\mathcal{F}_1(\omega_1 * \omega_2) = (\omega_1 \otimes \omega_2 \otimes \iota)(\Delta \otimes \iota)(W) \\ =&(\omega_1 \otimes \omega_2 \otimes \iota)W_{13}W_{23} = (\omega_1 \otimes \iota)(W)(\omega_2 \otimes \iota)(W). \end{aligned}$$

For $x \in L, \omega \in M_*$,

$$\mathcal{F}_p(\omega *^p l^p(x)) = \mathcal{F}_1(\omega * ({}_x\varphi)) = \mathcal{F}_1(\omega)\mathcal{F}_1({}_x\varphi) = \mathcal{F}_1(\omega)\mathcal{F}_p(l^p(x)).$$

Here, the first and last equality follows from commutativity of (5.32), (5.35) and (5.36). Since the range of l^p is dense in $L^p(M)_{\mathrm{left}}$, see Lemma 5.1.7, (4) follows.

□

5.8 The right Fourier transform and pairings

We show that a similar constuction as in Section 5.6 can be carried out to obtain a Fourier transform in terms of the right L^p-spaces. This makes the situation more symmetric than it appears now.

We also introduce a new pairing. The relation between pairings and Fourier transforms was already discussed in [43, Section 4]. We extend the results to the L^p-setting. The main result of this section is the introduction of a pairing between the spaces $L^p(M)$ and $L^p(\hat{M})$ $(1 \leq p \leq 2)$ that is compatible with respect to some compatible couple which we define later.

We will need some facts about right L^p-spaces. It follows that all the results of Section 5.5 have a right analogue. These analogues easily follow from their left versions using the following observation.

Lemma 5.8.1. *The anti-linear maps in the following diagram make the diagram commutative.*

$$\begin{tikzcd}
 & M_* \arrow[rrr, "\omega\mapsto\overline{\omega}"] \arrow[dr, "(5.8)"] & & & M_* \arrow[dr, "(5.10)"] & \\
L \arrow[ur, "l^1"] \arrow[r, "\Lambda"] \arrow[dr] \arrow[rrr, bend left, "x\mapsto x^*"] & \mathcal{H} \arrow[r, "(5.15)"] \arrow[rrr, bend right, "\xi\mapsto\overline{\xi}"] & R^* \arrow[rrr, bend left, "f(\cdot)\mapsto\overline{f(\cdot^*)}"] & R \arrow[ur, "r^1"] \arrow[r, "x\mapsto\overline{\Lambda(x^*)}"] \arrow[dr] & \overline{\mathcal{H}} \arrow[r, "(5.16)"] & L^*. \\
 & M \arrow[ur, "(5.9)"] \arrow[rrr, "x\mapsto x^*"] & & & M \arrow[ur, "(5.11)"] &
\end{tikzcd} \tag{5.37}$$

Proof. Note that the map $L \to R : x \mapsto x^*$ is well defined by Corollary 5.3.2. We check the commutativity. Any of the paths in the diagram departing from L

and ending in L^*, not passing R^* corresponds to one of the three computations below. For $x, y \in L$,

$$\begin{aligned}
\langle \overline{{}_x\varphi}, y\rangle_{L^*,L} &= \langle \varphi_{x^*}, y\rangle_{L^*,L} = \varphi(x^*y),\\
\langle \overline{\Lambda(x)}, y\rangle_{L^*,L} &= \langle \Lambda(y), \Lambda(x)\rangle = \varphi(x^*y),\\
\langle x^*, y\rangle_{L^*,L} &= {}_y\varphi(x^*) = \varphi(x^*y).
\end{aligned}$$

Since all outcomes are the same, 'the paths commute'. Any path in the diagram departing from either $M, \mathcal{H}, M_*$ on the left side and ending in L^* corresponds to one of the six computations below. For $x \in M, \xi \in \mathcal{H}, \omega \in M_*$ and $y \in L$,

$$\begin{array}{ll}
\overline{\langle x, y^*\rangle_{R^*,R}} = \overline{\varphi_{y^*}(x)} = \varphi(x^*y), & \langle x^*, y\rangle_{L^*,L} = {}_y\,\varphi(x^*) = \varphi(x^*y).\\
\langle \overline{\xi}, y\rangle_{L^*,L} = \langle \Lambda(y), \xi\rangle, & \overline{\langle \xi, y^*\rangle_{R^*,R}} = \overline{\langle \xi, \Lambda(y)\rangle} = \langle \Lambda(y), \xi\rangle.\\
\langle \overline{\omega}, y\rangle_{L^*,L} = \overline{\omega}(y), & \overline{\langle \omega, y^*\rangle_{R^*,R}} = \overline{\omega(y^*)} = \overline{\omega}(y).
\end{array}$$

Since, line by line, we have equality, the diagram commutes. Any other commutativity relation has already been checked. $\square$

Define $\mathcal{J} = \{\omega \mid \overline{\omega} \in \mathcal{I}\}$. For $\omega \in \mathcal{J}$, define the norm

$$\|\omega\|_{\mathcal{J}} = \max\{\|\omega\|, \|\xi(\overline{\omega})\|.$$

For $x \in \mathfrak{n}_\varphi^*$ define a norm by

$$\|x\|_{\mathfrak{n}_\varphi^*} = \max\{\|x\|, \|\Lambda(x^*)\|\}.$$

Now, the following properties follow from the commutativity of (5.37) and Section 5.5.

Corollary 5.8.2. *Consider right L^p-spaces. So every equality or inclusion should be understood within L^*.*

1. *We have $\mathcal{J} = \overline{\mathcal{H}} \cap M_*$. Within L^*, $\omega \in \mathcal{J}$ equals $\overline{\xi(\overline{\omega})}$.*
2. *The image of $L \to \mathcal{J} : x \mapsto_x \varphi$ is dense in $\mathcal{J}$ with respect to $\| \cdot \|_{\mathcal{J}}$.*
3. *We have $\mathfrak{n}_\varphi^* = \overline{\mathcal{H}} \cap M$. Within L^*, $x \in \mathfrak{n}_\varphi^*$ equals $\overline{\Lambda(x^*)}$. Moreover, let $\overline{R}$ be the closure of R in M, then also $\mathfrak{n}_\varphi^* = \overline{\mathcal{H}} \cap \overline{R}$.*
4. *R is dense in $\mathfrak{n}_\varphi^*$ with respect to $\| \cdot \|_{\mathfrak{n}_\varphi^*}$.*

Also a simple diagram chase yields the following.

Theorem 5.8.3.

1. *There exists a unitary map $\mathcal{F}_2^{\mathrm{right}} : \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ which is determined by $\overline{\Lambda(x)} \mapsto \overline{\hat{\Lambda}(\lambda(\varphi_x))}$, where $x \in R$.*

2. *There exists a bounded map $\mathcal{F}_1^{\mathrm{right}} : M_* \to \hat{M} : \omega \mapsto (\omega \otimes \iota)(W^*)$. Moreover, $\|\mathcal{F}_1^{\mathrm{right}}\| = 1$.*

3. *Let $p \in [1,2]$ and set q by $1/p + 1/q = 1$. There exists a unique bounded linear map $\mathcal{F}_p^{\mathrm{right}} : L^p(M)_{\mathrm{right}} \to L^q(\hat{M})_{\mathrm{right}}$ such that $\mathcal{F}_p^{\mathrm{right}}$ is compatible with $\mathcal{F}_1^{\mathrm{right}}$ and $\mathcal{F}_2^{\mathrm{right}}$ in the sense of Remark 5.1.3, i.e. the following diagram commutes:*

$$
\begin{tikzcd}
 & M_* \arrow[rrr, Rightarrow, "\mathcal{F}_1^{\mathrm{right}}"] \arrow[dr, "(5.10)"] & & & \hat{M} \arrow[dr, "(5.11)"] & \\
R \arrow[ur, "r^1"] \arrow[r, "r^p"] \arrow[dr, "x \mapsto \overline{\Lambda(x^*)}"'] & L^p(M)_{\mathrm{right}} \arrow[r] \arrow[rrr, Rightarrow, bend right, "\mathcal{F}_p^{\mathrm{right}}"] & L^* & \hat{R} \arrow[ur, "\hat{r}^1"] \arrow[r, "\hat{r}^q"] \arrow[dr, "x \mapsto \overline{\hat{\Lambda}(x^*)}"'] & L^q(\hat{M})_{\mathrm{right}} \arrow[r] & \hat{L}^* \\
 & \overline{\mathcal{H}} \arrow[ur, "(5.16)"'] \arrow[rrr, Rightarrow, "\mathcal{F}_2^{\mathrm{right}}"] & & & \overline{\mathcal{H}} \arrow[ur, "(5.16)"'] &
\end{tikzcd}
\tag{5.38}
$$

Moreover, $\|\mathcal{F}_p^{\mathrm{right}}\| \leq 1$.

We end this chapter by elaborating on pairings.

Theorem-Definition 5.8.4. *For $p \in [1,2]$, there is a pairing between $L^p(M)_{\mathrm{left}}$ and $L^p(\hat{M})_{\mathrm{right}}$, which is given by the following formula:*

$$
\begin{aligned}
\langle x, y \rangle_{L^p(M)_{\mathrm{left}}, L^p(\hat{M})_{\mathrm{right}}} &= \int \hat{\Phi}_q^{-1}(\mathcal{F}_p(x)) \cdot \hat{\Psi}_p^{-1}(y) d\hat{\phi} \\
&= \int \Phi_p^{-1}(x) \cdot \Psi_q^{-1}(\hat{\mathcal{F}}_p^{\mathrm{right}}(y)) d\hat{\phi}.
\end{aligned}
\tag{5.39}
$$

Proof. The equality of the two expressions on the right hand side follows from the following argument. Let $a, b \in \mathcal{T}_\varphi^2$. Then, using respectively Theorem 5.6.9 for the first equation and (3) of Proposition 5.4.6 for the second,

$$
\begin{aligned}
\int \hat{\Phi}_q^{-1}(\mathcal{F}_p(l^p(a))) \cdot \hat{\Psi}_p^{-1}(\hat{r}^p(b)) d\hat{\phi} = &\int [\lambda({}_a\varphi)\hat{d}^{\frac{1}{q}}] \cdot \hat{d}^{\frac{1}{p}} b \, d\hat{\phi} \\
= &\hat{\varphi}(b\lambda({}_a\varphi)) = ({}_a\varphi \otimes \hat{\varphi}_b)(W).
\end{aligned}
$$

Similarly (leaving the same technical issues as in the last computation to the reader), one may compute that

$$
\int \Phi_p^{-1}(x) \cdot \Psi_q^{-1}(\mathcal{F}_p^{\mathrm{right}})(y) d\phi = \int [a d^{\frac{1}{p}}] \cdot d^{\frac{1}{q}} (\iota \otimes \hat{\varphi}_b)(W) \, d\phi = ({}_a\varphi \otimes \hat{\varphi}_b)(W).
$$

Using the density result of Proposition 5.5.4 and Lemma 5.1.7 it follows that (5.39) holds for every $x \in L^p(M)_{\mathrm{left}}$ and $y \in L^p(\hat{M})_{\mathrm{right}}$. $\square$

In fact, it follows from the previous proof that for $x \in L$ and $y \in \hat{R}$,

$$
\langle l^p(x), r^p(y) \rangle_{L^p(M)_{\mathrm{left}}, L^p(\hat{M})_{\mathrm{right}}} = ({}_x\varphi \otimes \varphi_y)(W) = \langle \Lambda(x), \Lambda(y^*) \rangle. \tag{5.40}
$$

Remark 5.8.5. In the L^2-setting, we get the same pairing as in [43, Proposition 4.2]. In the L^1-setting, the pairing is well-known.

5.9 A distinguished choice for the interpolation parameter

Recall that in Sections 5.5 to 5.7 we considered the compatible couple (M, M_*) for the interpolation parameter $z = -1/2$, see Definition 5.2.1. For this parameter we are able to define a L^p-Fourier transform. In this section we show that the real part of the parameter is distinguished. More precisely, we investigate the example of $(M, \Delta) = SU_q(2)$ and show that given the fact that

$$\mathcal{F}_1 : M_* \to \hat{M} : \omega \mapsto (\omega \otimes \iota)(W), \tag{5.41}$$

is the L^1-Fourier transform the only interpolation parameters z that allows a passage to a L^p-Fourier transform are $z = -1/2 + it$, where $t \in \mathbb{R}$.

The importance of this result is strengthened by the final remark of [18]. For classical, locally compact groups there is an approximation property called Reiter's property (P_p), where $p \in [1, \infty)$. The definition assumes the existence of a net of functions in $L^p(G)$ satisfying the approximation axiom of [18, Definition 1.2]. Daws and Runde show that (P_1) and (P_2) can be defined for quantum groups as well and they use them to study (co-)amenability properties of quantum groups.

In the final remark of [18], Daws and Runde mention that it remains to be seen if there is a property (P_p) for any $p \in [1, \infty)$. In particular, they mention that it remains unclear how the L^p-space associated with a quantum group should be turned into a L^1-module. In [31] this is done using Izumi's L^p-spaces for the complex interpolation parameter $z = -1/2$, whereas in [17] a similar, but not identical construction was used for the parameter $z = 0$. We believe that the Fourier transform indicates that the most natural choice would be $z = -1/2$, or from the considerations in Section 5.8 $z = 1/2$.

Recall the notational conventions from Section 5.2.

Theorem 5.9.1. *Consider $(M, \Delta) = SU_q(2)$ and let $z, z' \in \mathbb{C}$. Let $\mathcal{F}_1 : M_* \to \hat{M}$ be defined as in (5.41). Suppose that there is bounded map $F_2 : L^2_{(z)}(M) \to$*

$L^2_{(z')}(\hat{M})$ making the following diagram commutative

$$\begin{tikzcd}
 & M_* \arrow[rrrr, Rightarrow, "\mathcal{F}_1"] \arrow[drr, "(i^\infty_{(-z)})^*"] & & & & \hat{M} \arrow[drr, "(\hat{i}^1_{(-z')})^*"] & & \\
L_{(z)} \arrow[ur, "i^1_{(z)}"] \arrow[r, "i^2_{(z)}"] & L^2_{(z)}(M) \arrow[rr, "\subseteq"] \arrow[rrrr, Rightarrow, bend right, "F_2"'] & & L^*_{(-z)} & \hat{L}_{(z')} \arrow[ur, "\hat{i}^\infty_{(z')}"] \arrow[r, "\hat{i}^2_{(z')}"] & L^2_{(z')}(\hat{M}) \arrow[rr, "\subseteq"] & & \hat{L}^*_{(-z')}.
\end{tikzcd} \tag{5.42}$$

Then $z = -1/2 + it$ for some $t \in \mathbb{R}$.

Proof. We will prove that F_2 is unbounded unless $z = -1/2 + it$ for some $t \in \mathbb{R}$. We need three preparations.

Firstly, the modular automorphism group of φ is given by the formula

$$\sigma_t(x) = (\gamma\gamma^*)^{it} x (\gamma\gamma^*)^{-it}, \qquad t \in \mathbb{R}.$$

Hence it follows from (1.11) that $\alpha \in \mathcal{T}_\varphi$ and

$$\sigma_z(\alpha) = q^{-2iz}\alpha, \qquad \sigma_z(\alpha^*) = q^{2iz}\alpha^*.$$

Secondly, for $a, b \in \mathcal{T}_\varphi, x \in \mathcal{T}^2_\varphi, z \in \mathbb{C}$, we find

$$\begin{aligned}
\varphi^{(z)}_x(a^*b) =& \langle xJ\nabla^{\bar{z}}\Lambda(a), J\nabla^{-z}\Lambda(b)\rangle = \langle \nabla^{z+\frac{1}{2}} x \nabla^{-z-\frac{1}{2}} \nabla J \nabla^{1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle \\
=& \langle \sigma_{-i(z+\frac{1}{2})}(x)\Lambda(\sigma_{-i}(a^*)), \Lambda(b^*)\rangle = \varphi(b\sigma_{-i(z+\frac{1}{2})}(x)\sigma_{-i}(a^*)) \\
=& \varphi(a^*b\sigma_{-i(z+\frac{1}{2})}(x)).
\end{aligned}$$

So we conclude that $\varphi^{(z)}_x = \sigma_{-i(z+\frac{1}{2})}(x)\varphi$.

Thirdly, we identify $\hat{M}$ with $\oplus_{l \in \frac{1}{2}\mathbb{N}} M_{2l+1}(\mathbb{C})$. Let $e^{(n/2)}_{n/2,n/2}$, and respectively $e^{(n/2)}_{-n/2,-n/2}$, be the element of $\hat{M}$, with matrix elements equal to zero everywhere, except for the summand with index $n/2$, where it has a 1 on the upper left, respectively lower right, corner. Using the Peter-Weyl orthogonality relations and recalling Lemmas 1.7.4 and 1.7.3, we see that for every $n \in \mathbb{N}$:

$$\begin{aligned}
(\varphi \otimes \iota)(W(\alpha^n \otimes 1)) =& \bigoplus_{l \in \frac{1}{2}\mathbb{N}} (\varphi \otimes \iota)(t^{(l)}(t^{(n/2)*}_{-n/2,-n/2} \otimes 1)) = \varphi((\alpha^*)^n\alpha^n)e^{(n/2)}_{-n/2,-n/2}, \\
(\varphi \otimes \iota)(W((\alpha^*)^n \otimes 1)) =& \bigoplus_{l \in \mathbb{N}} (\varphi \otimes \iota)(t^{(l)}(t^{(n/2)*}_{n/2,n/2} \otimes 1)) = \varphi(\alpha^n(\alpha^*)^n)e^{(n/2)}_{n/2,n/2},
\end{aligned} \tag{5.43}$$

$$\varphi((\alpha^*)^n\alpha^n) = \frac{(1-q^2)q^{2n}}{1-q^{2n+2}}, \qquad \varphi(\alpha^n(\alpha^*)^n) = \frac{(1-q^2)}{1-q^{2n+2}}.$$

By (1.14) and the fact that D is diagonal, see Lemma 1.7.4, we see that the elements in (5.43) are in $\mathcal{T}^2_{\hat{\varphi}}$ and

$$\hat{\sigma}_z(e^{(n/2)}_{-n/2,-n/2}) = e^{(n/2)}_{-n/2,-n/2}, \qquad \hat{\sigma}_z(e^{(n/2)}_{n/2,n/2}) = e^{(n/2)}_{n/2,n/2}. \tag{5.44}$$

Now we prove that F_2 must be unbounded by proving that the map given by $U_{(z')}F_2U^*_{(z)} : \mathcal{H} \to \mathcal{H}$ is unbounded. Recall that $U_{(z)}$ is defined in Proposition 5.4.6.

$$\begin{aligned} &F_2U^*_{(z)}\Lambda(\sigma_{-i(z+1/2)/2}(\alpha^n)) = F_2 i^2_{(z)}(\alpha^n) = (\hat{i}^1_{(-z')})^* F_1 i^1_{(z)}(\alpha^n) \\ =&(\hat{i}^1_{(-z')})^*(\varphi^{(z)}_{\alpha^n} \otimes \iota)(W) = (\hat{i}^1_{(-z')})^*(\varphi \otimes \iota)(W(\sigma_{-i(z+1/2)}(\alpha^n) \otimes 1)) \\ =&q^{-2n(z+1/2)}(\hat{i}^1_{(-z')})^*(\varphi \otimes \iota)(W(\alpha^n \otimes 1)). \end{aligned}$$

Since $(\varphi\otimes\iota)(W(\alpha^n\otimes 1)) \in \mathcal{T}^2_{\hat{\varphi}}$, we see that by commutativity of the right triangle in (5.42),

$$F_2U^*_{(z)}\Lambda(\sigma_{-i(z+1/2)/2}(\alpha^n)) = q^{-2n(z+1/2)}(\hat{i}^2_{(z')})(\varphi \otimes \iota)(W(\alpha^n \otimes 1)).$$

Hence,

$$\begin{aligned} &U_{(z')}F_2U^*_{(z)}\Lambda(\sigma_{-i(z+1/2)/2}(\alpha^n)) \\ =&q^{-2n(z+1/2)}\hat{\Lambda}(\hat{\sigma}_{-i(z'+1/2)/2}(\varphi \otimes \iota)(W(\alpha^n \otimes 1))) \\ \overset{(5.43),(5.44)}{=}&q^{-2n(z+1/2)}\hat{\Lambda}((\varphi \otimes \iota)(W(\alpha^n \otimes 1))) \\ =&q^{-2n(z+1/2)}\xi(\alpha^n\varphi) = q^{-2n(z+1/2)}\Lambda(\alpha^n). \end{aligned}$$

Hence,

$$U_{(z')}F_2U^*_{(z)} : \Lambda(\alpha^n) = q^{n(z+1/2)}\Lambda(\sigma_{-i(z+1/2)/2}(\alpha^n)) \mapsto q^{-n(z+1/2)}\Lambda(\alpha^n).$$

which is unbounded in case $\mathrm{Re}(z) > -1/2$.

By a similar computation, one finds that

$$U_{(z')}F_2U^*_{(z)} : \Lambda((\alpha^*)^n) \mapsto q^{n(z+1/2)}\Lambda((\alpha^*)^n).$$

In this case we see that $U_{(z')}F_2U^*_{(z)}$ is unbounded for $\mathrm{Re}(z) < -1/2$. □

Remark 5.9.2. By Pontrjagin duality also the dual statement holds. So in order to get a proper Fourier theory on quantum groups with Fourier transforms and inverse Fourier transforms, one is obliged to take the interpolation parameter on both M and $\hat{M}$ to be $-1/2$.

Remark 5.9.3. We believe that Reiter's property (P_p) for $p \in [1,2]$ [18] can be defined using both the complex interpolation parameters $z = -1/2$ and $z = 1/2$. The reason for this believe is that definitions of (P_1) and (P_2) coincide on the intersection for this choice of parameters. However, we did not investigate this any further and we do not know if this can be done properly in the setting of operator spaces. Moreover, it is even more uncertain if this is going to deliver a proof of the equivalence of (P_1) and (P_2), which in turn would prove the long standing question of whether or not amenability of a quantum group is equivalent to co-amenability of its dual.

Chapter 6

Walsh basis for L^p-spaces of hyperfinite III_λ factors, $0 < \lambda \leq 1$

In the present chapter we study the non-commutative L^p-spaces associated with the hyperfinite factors of type III_λ, where $0 < \lambda \leq 1$. In particular, we are interested in the decomposition of this space in terms of L^p-spaces of matrix algebras and the construction of a very classical Schauder basis, namely the *Walsh system*.

Recall that the classical Walsh system is defined as follows. One firstly defines the Rademacher functions:

$$r_j(x) = \mathrm{sgn}\left(\sin\left(2^j \pi x\right)\right), \qquad j \in \mathbb{N}, x \in [0,1].$$

The *classical Walsh system*, see e.g. [45], is defined as the sequence of functions given by:

$$w_n = \prod_{\gamma_i \neq 0} r_i, \qquad \text{where } n = \sum_{i=0}^{\infty} \gamma_i 2^i, \gamma_i \in \{0,1\}. \tag{6.1}$$

It is a classical result that the sequence $(w_n)_{n\in\mathbb{N}}$ forms a Schauder basis in the spaces $L^p([0,1],\mu)$ for every $1 < p < \infty$, see [45, Theorem IV.15]. Here μ denotes the Lebesgue measure.

Proper non-commutative generalizations of the Walsh system have been found for the L^p-spaces associated with the hyperfinite II_1 and II_∞ factor [28]. Also, related problems have been studied in [23], [27], where non-commutative trigonometric systems and non-commutative Vilenkin systems where constructed. Furthermore, in [70] a non-commutative Haar system was built for hyperfinite type III_λ factors, $0 < \lambda \leq 1$.

Here we continue this line by constructing a Walsh system for the hyperfinite III$_\lambda$-factors, where $0 < \lambda \leq 1$. We elaborate on the special commutative case $L^p([0,1], \mu_\alpha)$. Here, μ_α is the Lebesgue measure in case $\alpha = \frac{1}{2}$. In case $\alpha \neq \frac{1}{2}$, the measure μ_α is a biased measure which is singular to the Lebesgue measure and appears naturally in the construction of III$_\lambda$ factors, c.f. [44].

The structure of this chapter is as follows. Section 6.1 recalls the necessary results on general non-commutative L^p-spaces. In Section 6.2 we introduce the hyperfinite III$_\lambda$ factors and fix notation. Section 6.3 contains our main result, which is the construction of a non-commutative Walsh system as a Schauder basis in the L^p-spaces associated with the hyperfinite III$_\lambda$ factors, $1 < p < \infty, 0 < \lambda < 1$. In Section 6.4 we construct a Walsh system for the hyperfinite III$_1$ factor. Finally, we make remarks on the classical case in Section 6.5.

The contents of this chapter are contained in the joint preprint with D. Potapov and F. Sukochev [11].

6.1 Non-commutative L^p-spaces and expectations

As mentioned in Chapter 5, non-commutative L^p-spaces appear in different guises. To keep this chapter self-contained, we briefly recall one of the definitions of L^p-spaces. We choose for the abstract construction by means of complex interpolation, since this is the most useful one for our purposes. We also introduce projections associated with conditional expectation values on von Neumann subalgebras.

For the details on the complex interpolation method, we refer to Section 5.1 or [3]. Let M be a von Neumann algebra with faithful, normal state ω. We consider the non-dotted part of the (commutative) diagram:

$$\begin{tikzcd}
& M_* \arrow[dr, hook, "\omega\mapsto\omega"] & \\
M \arrow[ur, hook, "x\mapsto x\omega"] \arrow[r, dashed, hook, "i^p_{(-\frac{1}{2})}"] \arrow[dr, hook, "x\mapsto x"'] & (M, M_*)_{[\frac{1}{p}]} \arrow[r, dashed, hook] & M_*. \\
& M \arrow[ur, hook, "x\mapsto x\omega"'] &
\end{tikzcd} \tag{6.2}$$

This turns the pair (M, M_*) into a compatible couple of Banach spaces, see Definition 5.1.1 or [3, Section 2.3]. The complex interpolation method at parameter $\frac{1}{p}$ gives by definition a Banach space $(M, M_*)_{[\frac{1}{p}]}$ which is a subset of M_*, where the inclusion is a norm-decreasing map. Moreover, the complex interpolation method gives an embedding:

$$i^p_{(-\frac{1}{2})} : M \to (M, M_*)_{[\frac{1}{p}]}.$$

See also the dotted part of (6.2). It is proved in [54] that the Banach space $(M, M_*)_{[\frac{1}{p}]}$ is isometrically isomorphic to the non-commutative L^p-spaces associated with M as were defined by Haagerup and Connnes/Hilsum. An elaborate discussion on this can be found in Chapter 5 or the papers [40], [54], [77], [78]. In particular, the construction is up to an isometric isomorphism independent of the choice of ω. Recall that we set $L^p(M)_{\rm left} = (M, M_*)_{[\frac{1}{p}]}$ as the non-commutative L^p-space associated with M. The norm on $L^p(M)_{\rm left}$ will be denoted by $\| \cdot \|_p$.

Remark 6.1.1. We have an *equality* of Banach spaces $L^1(M)_{\rm left} = M_*$, see [3, Theorem 4.2.2]. By the same argument M is isometrically isomorpic to $L^\infty(M)_{\rm left}$ via the embedding i^∞.

Remark 6.1.2. Recall that the L^p-spaces we defined are also called L^p-spaces with respect to the *left* injection. If one changes both the embeddings $M \hookrightarrow M_*$ in (6.2) by $x \mapsto \omega x$, the interpolated spaces are isometrically isomorphic to the present L^p-spaces and we refer to this construction as L^p-spaces with respect to the *right* injection. Other injections have been given in [54], see also Chapter 5. However, the constructions in the present chapter only work for the left injection and in slightly different form also for the right injection. We comment on this when it feels appropriate. Unless stated otherwise, every L^p-space should be understood with respect to the left injection.

Suppose that N is a von Neumann subalgebra of M such that there exists a ω-preserving conditional expectation value $\mathbb{E} : M \to N$ [75, Definition IX.4.1]. Denote the inclusion by $j : N \to M$. Let $\mathbb{E}' : M_* \to N_* : \omega \mapsto \omega|_N$ be the restriction map and also consider the extension map $j' : N_* \to M_* : \omega \mapsto \omega \circ \mathbb{E}$. Note that,

$$(\mathbb{E}(x)\omega)(y) = \omega(y\mathbb{E}(x)) = \omega(yx) = (x\omega)(y), \qquad x \in M, y \in N, \tag{6.3}$$
$$(x\omega)(y) = \omega(yx) = \omega(\mathbb{E}(y)x) = (x\omega)(\mathbb{E}(y)), \qquad x \in N, y \in M. \tag{6.4}$$

It follows from (6.3) that the pair given by $\mathbb{E}$ and $\mathbb{E}'$ forms a morphism in the category of compatible couples of Banach spaces [3, Section 2.3] or Remark 5.1.3. By complex interpolation, we obtain a norm-decreasing map:

$$\mathbb{E}^p : L^p(M)_{\rm left} \to L^p(N)_{\rm left}, \qquad 1 \leq p \leq \infty. \tag{6.5}$$

It follows from (6.4) that the pair given by j and j' forms a morphism in the category of compatible couples of Banach spaces [3, Section 2.3] or Remark 5.1.3. Complex interpolation yields a norm-decreasing map:

$$j^p : L^p(N)_{\rm left} \to L^p(M)_{\rm left}.$$

In fact, j^p is isometric, since

$$\|x\|_p = \|\mathbb{E}^p \circ j^p(x)\|_p \leq \|j^p(x)\|_p \leq \|x\|_p, \qquad x \in \mathcal{L}^p(N).$$

Hence, we may *identify* $L^p(N)_{\rm left}$ as a complemented closed subspace of the space $L^p(M)_{\rm left}$.

Remark 6.1.3. Recall from Proposition 5.7.2 that also *left* multiplication is compatible with respect to (6.2), i.e.

$$x(y\omega) = (xy)\omega, \qquad x, y \in M.$$

Therefore, for every $x \in M$, we can interpolate left multiplication with x to give a bounded map m_x^p determined by

$$m_x^p : L^p(M)_{\mathrm{left}} \to L^p(M)_{\mathrm{left}} : i^p_{(-\frac{1}{2})}(y) \mapsto i^p_{(-\frac{1}{2})}(xy), \qquad y \in M. \tag{6.6}$$

For $x \in M$ and $y \in L^p(M)_{\mathrm{left}}$, we conveniently write xy for $m_x^p(y)$.

We recall the necessary results on *martingales*. Let M be a von Neumann algebra with faithful, normal state ω as in Section 6.1. Let $(M_s)_{s\in\mathbb{N}}$ be an increasing filtration of von Neumann subalgebras of M such that their union is σ-weakly dense in M. Suppose that there exist ω-preserving conditional expectation values $\mathbb{E}_s : M \to M_s$. Define $\mathbb{D}_s = \mathbb{E}_s - \mathbb{E}_{s-1}$. As explained, we get a sequence of complemented closed subspaces of $L^p(M)_{\mathrm{left}}$,

$$L^p(M_0)_{\mathrm{left}} \subseteq L^p(M_1)_{\mathrm{left}} \subseteq L^p(M_2)_{\mathrm{left}} \subseteq \ldots \subseteq L^p(M)_{\mathrm{left}},$$

with projections $\mathbb{E}_s^p : L^p(M)_{\mathrm{left}} \to L^p(M_s)_{\mathrm{left}}$ and differences $\mathbb{D}_s^p = \mathbb{E}_s^p - \mathbb{E}_{s-1}^p$.

A *L^p-martingale* with respect to $(M_s)_{s\in\mathbb{N}}$ is a sequence $(x_s)_{s\in\mathbb{N}}$ with $x_s \in L^p(M)_{\mathrm{left}}$ and $\mathbb{E}_s^p(x_{s+1}) = x_s$. In particular $x_s \in L^p(M_s)_{\mathrm{left}}$ and $x_s - x_{s-1} = \mathbb{D}_s^p(x_s)$. A L^p-martingale $(x_s)_{s\in\mathbb{N}}$ is *finite* if there is a $n \in \mathbb{N}$ such that for all $s \geq n$ we have $\mathbb{D}_s^p(x_s) = 0$. If $x \in L^p(M)_{\mathrm{left}}$, then the sequence $(\mathbb{E}_s^p(x))_{s\in\mathbb{N}}$ is a L^p-martingale. Such sequences are called *bounded* L^p-martingales. Note that the original definition of bounded is different, see [38, Remark 6.1]. It follows that finite L^p-martingales are bounded.

The following theorem follows from the Burkholder-Gundy inequalities, as first proved in the present setting in [41]. The theorem also appears in [38], where the notation is closer to ours.

Theorem 6.1.4 (Theorem 6.3 of [38]). *Let $1 < p < \infty$. There exists a constant C_p, such that for every finite L^p-martingale $(x_s)_{s\in\mathbb{N}}$ and every choice of signs $\varepsilon_s \in \{-1, 1\}$,*

$$\| \sum_{s=0}^{\infty} \varepsilon_s \mathbb{D}_s^p(x_s) \|_p \leq C_p \| \sum_{s=0}^{\infty} \mathbb{D}_s^p(x_s) \|_p.$$

It follows directly that the statement holds for every bounded L^p-martingale.

6.2 The setup: non-commutative L^p-spaces associated with the hyperfinite factors

In this section, we fix the notation for the rest of this chapter. We introduce hyperfinite factors as the direct limit of matrix algebras. The results in this section can be found in [75] and [76].

We define the following matrix algebras:

$$N_s = \begin{cases} M_2(\mathbb{C})^{\otimes \frac{s+1}{2}} & \text{for } s \in 2\mathbb{N}+1, \\ M_2(\mathbb{C})^{\otimes \frac{s}{2}} \bigotimes \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} & \text{for } s \in 2\mathbb{N}. \end{cases}$$

For $s \in 2\mathbb{N}+1$, we consider N_s as a subalgebra of N_{s+1} by means of the embedding $x \mapsto x \otimes 1$. For $s \in 2\mathbb{N}$, there is a natural inclusion $N_s \subseteq N_{s+1}$. Fix $0 < \alpha \leq \frac{1}{2}$ and let:

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}, \qquad A_n = A_1^{\otimes n}, n \in \mathbb{N}.$$

Define a state ρ_s on N_s by setting

$$\rho_s(x) = \mathrm{Tr}(xA_{\lceil \frac{s+1}{2} \rceil}), \qquad x \in N_s.$$

Here $\lceil \frac{s+1}{2} \rceil$ is the smallest integer that is greater than or equal to $\frac{s+1}{2}$.

We let R_α be the von Neumann algebra given by the infinite tensor product of $M_2(\mathbb{C})$ equipped with the states ρ_1, see [76, Section XVIII.1]. Then, R_α is a type III$_\lambda$ factor where $\lambda = \frac{\alpha}{1-\alpha}$ in case $0 < \alpha < \frac{1}{2}$ and R_α is factor of type II$_1$ in case $\alpha = \frac{1}{2}$. We have natural injective $*$-homomorphisms

$$\pi_s : N_s \to R_\alpha, \qquad s \in \mathbb{N}.$$

Furthermore, there is a distinguished faithful normal state ρ_α on R_α, which is characterized by the property:

$$\rho_\alpha(\pi_s(x)) = \rho_s(x), \qquad s \in \mathbb{N},\ x \in N_s. \tag{6.7}$$

Moreover, we have the following lemma, which is well known.

Lemma 6.2.1. *For every $s \in \mathbb{N}$ the following holds.*

1. *The embedding $N_s \to N_{s+1}$ carries to the inclusion $\pi_s(N_s) \subseteq \pi_{s+1}(N_{s+1})$.*
2. *The modular automorphism group σ^{ρ_α} leaves $\pi_s(N_s)$ globally invariant, i.e. $\sigma^{\rho_\alpha}(\pi_s(N_s)) = \pi_s(N_s)$.*
3. *The union $\cup_{s\in\mathbb{N}}\pi_s(N_s)$ is σ-weakly dense in R_α.*

For convenience of notation, we will *identify* N_s with its image under π_s, so that N_s is a von Neumann subalgebra of R_α. By (6.7) we see that ρ_s is the restriction of ρ_α to N_s. Property (2) of Lemma 6.2.1 implies that there is a ρ_α-preserving conditional expectation value, c.f. [75, Theorem IX.4.2]. From now on, we use the following notation for this map:

$$\mathbb{E}_s : R_\alpha \to N_s. \tag{6.8}$$

In addition, we set $N_{-1} = \mathbb{C}1$, the one-dimensional subalgebra generated by the unit of R_α. We set $\mathbb{E}_{-1} : R_\alpha \to N_{-1}$ as the corresponding ρ_α-preserving conditional expectation value, which in fact is given by the map ρ_α.

It follows from the preliminaries in Section 6.1 that we get non-commutative L^p-spaces $L^p(N_s)_{\mathrm{left}}$, with respect to the faithful, normal state ρ_s, with $s \in \mathbb{N}$. Similarly, we will use the notation $L^p(R_\alpha)_{\mathrm{left}}$ for the L^p-space associated with R_α with respect to ρ_α. As explained, we *identify* $L^p(N_s)_{\mathrm{left}}$ as a closed subspace of $L^p(N_{s+1})_{\mathrm{left}}$. Similarly, we may *identify* $L^p(N_s)_{\mathrm{left}}$ as a closed subspace of $L^p(R_\alpha)_{\mathrm{left}}$ and we get a chain of closed subspaces,

$$L^p(N_0)_{\mathrm{left}} \subseteq L^p(N_1)_{\mathrm{left}} \subseteq L^p(N_2)_{\mathrm{left}} \subseteq \ldots \subseteq L^p(R_\alpha)_{\mathrm{left}}, \qquad 1 \leq p \leq \infty. \tag{6.9}$$

As a vector space N_s is isomorphic to $\mathcal{L}^p(N_s)$ by means of the mapping $i^p_{(-\frac{1}{2})}$, see (6.2). For $x \in N_s$, the norm of $i^p_{(-\frac{1}{2})}(x) \in \mathcal{L}^p(N_s)$ may be directly computed as

$$\|i^p_{(-\frac{1}{2})}(x)\|_p = \mathrm{Tr}(|xA_s^{\frac{1}{p}}|^p)^{\frac{1}{p}},$$

see the discussions in Chapter 5 or [70, Remark 3.1].

Interpolating the conditional expectation values $\mathbb{E}_s$, we find projections

$$\mathbb{E}_s^p : L^p(R_\alpha)_{\mathrm{left}} \to L^p(N_s)_{\mathrm{left}}.$$

We will need the following approximation result.

Proposition 6.2.2 (Theorem 8 of [33]). *For $1 \leq p < \infty$ and $x \in L^p(R_\alpha)_{\mathrm{left}}$,*

$$\|x - \mathbb{E}_s^p(x)\|_p \to 0, \qquad \text{as } s \to \infty. \tag{6.10}$$

In particular, for $1 \leq p < \infty$, the union $\cup_{s\in\mathbb{N}} L^p(N_s)_{\mathrm{left}}$ is dense in $L^p(R_\alpha)_{\mathrm{left}}$.

6.3 Non-commutative Walsh system

Let $\mathcal{X}$ be a (complex) Banach space. Recall that a sequence $\mathrm{x} = (x_i)_{i\in\mathbb{N}}$ in $\mathcal{X}$ is called a *Schauder basis* if for every $x \in \mathcal{X}$ there are unique scalars $\alpha_i \in \mathbb{C}$ such that $x = \sum_{i=0}^\infty \alpha_i x_i$. In fact, x will form a Schauder basis of $\mathcal{X}$ if and only if the linear span of $x_i, i \in \mathbb{N}$ is dense in $\mathcal{X}$ and there is a constant C such that for every choice of scalars $\alpha_i \in \mathbb{C}$ and every $n, m \in \mathbb{N}$ with $n > m$,

$$\|\sum_{i=0}^m \alpha_i x_i\|_{\mathcal{X}} \leq C \|\sum_{i=0}^n \alpha_i x_i\|_{\mathcal{X}}. \tag{6.11}$$

The constant C is also called the *basis constant* [63, Section 1.a].

In [28], a non-commutative *Walsh system* was given for the L^p-spaces associated with the hyperfinite II$_1$-factor $R_{\frac{1}{2}}$ for $1 < p < \infty$. Recall that the system is constructed as follows. Consider the matrices:

$$\begin{aligned} r^{(0,0)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r^{(1,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ r^{(0,1)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad r^{(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{6.12}$$

For $n \in \mathbb{N}$ we consider the binary decomposition $n = \sum_{i=0}^{\infty} \gamma_i 2^i$, where $\gamma_i \in \{0,1\}$. We define:

$$w_n = \bigotimes_{i=0}^{\infty} r^{(\gamma_{2i}, \gamma_{2i+1})}. \tag{6.13}$$

The sequence $\mathbf{w} = (w_n)_{n\in\mathbb{N}}$ is called the *Walsh system*. For $\alpha = \frac{1}{2}$, the state $\rho_{\frac{1}{2}}$ is a trace and $R_{\frac{1}{2}}$ is the hyperfinite II_1-factor. In that case, it is well-known that $L^p(R_{\frac{1}{2}})_{\mathrm{left}}$ is isometrically isomorphic to the semi-finite L^p-spaces with respect to the trace $\rho_{\frac{1}{2}}$, see also [37, Section 2]. Recall that the latter space can be defined as the completion of $R_{\frac{1}{2}}$ with respect to the norm $\|x\|_p = \rho_{\frac{1}{2}}(|x|^p)^{\frac{1}{p}}$.

Theorem 6.3.1 (Proposition 5 of [28]). *For $1 < p < \infty$, the Walsh system $\mathbf{w}$ forms a Schauder basis in the semi-finite L^p-spaces associated with the trace $\rho_{\frac{1}{2}}$ on $R_{\frac{1}{2}}$.*

In the present chapter, we extend the result to the hyperfinite factors R_α. Considering the interpolation structure as described in Section 6.1, we can consider the Walsh system $\mathbf{w}$ as a sequence in $L^p(R_\alpha)_{\mathrm{left}}$ by means of the embedding $i^p_{(-\frac{1}{2})}$, see (6.2). We prove that $\mathbf{w}$ is a Schauder basis in $L^p(R_\alpha)_{\mathrm{left}}$ for $1 < p < \infty$.

Remark 6.3.2. Note that we do not incorporate p explicitly in the notation of our basis $\mathbf{w}$. To justify this, note that by definition $L^p(R_\alpha)_{\mathrm{left}}$ is as a set a subset of $(R_\alpha)_*$, though their norms are different of course. Also $R_\alpha \simeq i^\infty_{(-\frac{1}{2})}(R_\alpha)$ is identified as a subset of $(R_\alpha)_*$ by means of (6.2). As an element of $(R_\alpha)_*$, the definition of $\mathbf{w}$ does not depend on p. The principle of this slight abuse of notation is comparable to the fact that one does not distinguish a classical Walsh function (6.1) as an element of $L^p([0,1],\mu)$ for different p.

We fix some auxiliary notation. Let $s \in \mathbb{N}$. Recall that $\mathbb{E}_s : R_\alpha \to N_s$ was defined in (6.8). Put

$$\mathbb{D}_s = \mathbb{E}_s - \mathbb{E}_{s-1},$$

and set $\mathcal{U}_s = \mathbb{D}_s(R_\alpha)$. Note that $\mathcal{U}_s \subseteq N_s$. Moreover,

$$\begin{aligned} \mathcal{U}_s &= \mathrm{span}\left\{ w_n \mid 2^s \leq n < 2^{s+1} \right\} \\ &= \begin{cases} \mathrm{span}\{M_2(\mathbb{C})^{\otimes \frac{s}{2}} \otimes r^{(1,0)}\}, & \text{if } s \in 2\mathbb{N}, \\ \mathrm{span}\{M_2(\mathbb{C})^{\otimes \frac{s-1}{2}} \otimes r^{(0,1)}, M_2(\mathbb{C})^{\otimes \frac{s-1}{2}} \otimes r^{(1,1)}\}, & \text{if } s \in 2\mathbb{N}+1. \end{cases} \end{aligned}$$

Define the *Rademacher* matrices:

$$r_s = \begin{cases} \left(\bigotimes_{i=1}^{\frac{s}{2}} 1\right) \otimes r^{(1,0)} \in N_s, & \text{if } s \in 2\mathbb{N}, \\ \left(\bigotimes_{i=1}^{\frac{s-1}{2}} 1\right) \otimes r^{(0,1)} \in N_s, & \text{if } s \in 2\mathbb{N}+1. \end{cases}$$

In particular, $r_s \in \mathcal{U}_s$.

Lemma 6.3.3. *For $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$, we have*

$$w_{n-2^k} = w_n r_k = \varepsilon\, r_k w_n. \tag{6.14}$$

Here, $\varepsilon \in \{-1, 1\}$ is positive, unless k is odd and $2^k + 2^{k-1} \leq n < 2^{k+1}$.

Proof. Suppose that $k \in 2\mathbb{N}$. Then,

$$w_n = \left(\bigotimes_{i=0}^{\frac{k}{2}-1} r^{(\gamma_{2i},\gamma_{2i+1})}\right) \otimes r^{(1,0)} \qquad \text{and} \qquad r_k = \left(\bigotimes_{i=0}^{\frac{k}{2}-1} 1\right) \otimes r^{(1,0)}.$$

Hence, $w_n r_k = r_k w_n = \bigotimes_{i=0}^{\frac{k}{2}-1} r^{(\gamma_{2i},\gamma_{2i+1})}$. Taking into account that the binary decomposition of n and $n - 2^k$ are the same except for the k-th digit, we see that $w_n r_k = r_k w_n = w_{n-2^k}$.

Now, consider the case $k \in 2\mathbb{N}+1$. If $2^k \leq n < 2^k + 2^{k-1}$, then

$$w_n = \left(\bigotimes_{i=0}^{\frac{k-3}{2}} r^{(\gamma_{2i},\gamma_{2i+1})}\right) \otimes r^{(0,1)} \qquad \text{and} \qquad r_k = \left(\bigotimes_{i=0}^{\frac{k-3}{2}} 1\right) \otimes r^{(0,1)}.$$

It follows again that $w_n r_k = r_k w_n = w_{n-2^k}$. If $2^k + 2^{k-1} \leq n < 2^{k+1}$, then

$$w_n = \left(\bigotimes_{i=0}^{\frac{k-3}{2}} r^{(\gamma_{2i},\gamma_{2i+1})}\right) \otimes r^{(1,1)} \qquad \text{and} \qquad r_k = \left(\bigotimes_{i=0}^{\frac{k-3}{2}} 1\right) \otimes r^{(0,1)}.$$

Using the fact that $r^{(1,1)} r^{(0,1)} = -r^{(0,1)} r^{(1,1)} = r^{(1,0)}$ we now get $w_n r_k = -r_k w_n = w_{n-2^k}$. □

Let $\mathbb{P}_n : \cup_{i=0}^{\infty} N_i \to R_\alpha$ be the projection determined by

$$\mathbb{P}_n\left(\sum_{i=0}^{m} \alpha_i w_i\right) = \sum_{i=0}^{n} \alpha_i w_i, \qquad m > n, \alpha_i \in \mathbb{C}.$$

Note that directly after the next proposition we extend the domain of $\mathbb{P}_n$ to R_α, c.f. Remark 6.3.5.

Theorem 6.3.4. *Fix $x = \sum_{i=0}^{m} \alpha_i w_i \in R_\alpha$ with $\alpha_i \in \mathbb{C}$. For every $n < m$:*

$$w_n \mathbb{P}_n(x) = \mathbb{E}_{-1}(w_n x) + \sum_{i \text{ with } \gamma_i = 1} \mathbb{D}_i(w_n x), \tag{6.15}$$

where $\gamma_i \in \{0,1\}$ are such that $n = \sum_{i=0}^{\infty} \gamma_i 2^i$.

Proof. The proof proceeds by induction to n. For $n = 0$, note that the summation on the right hand side of (6.15) vanishes. We find:

$$w_0 \mathbb{P}_0(x) = \alpha_0 w_0 = \mathbb{E}_{-1}(w_0 x).$$

Now, suppose that (6.15) holds for all numbers stricly smaller than n. Let k be such that $2^k \leq n < 2^{k+1}$, so that $w_n \in \mathcal{U}_k$. Write $n' = n - 2^k$. Then, by (6.14) we find,

$$w_n \mathbb{P}_n(x) = w_n \sum_{i=0}^{n} \alpha_i w_i = w_{n'} r_k \left(\sum_{i=0}^{2^k - 1} \alpha_i w_i \right) + w_{n'} r_k \left(\sum_{i=2^k}^{n} \alpha_i w_i \right). \tag{6.16}$$

For the left summation on the right hand side, the appearance of the Rademacher r_k ensures that $w_{n'} r_k \left(\sum_{i=0}^{2^k-1} \alpha_i w_i \right) \in \mathcal{U}_k$. Hence,

$$w_{n'} r_k \left(\sum_{i=0}^{2^k - 1} \alpha_i w_i \right) = \mathbb{D}_k \left(w_{n'} r_k \left(\sum_{i=0}^{2^k - 1} \alpha_i w_i \right) \right). \tag{6.17}$$

By (6.14) we have $r_k w_i \notin \mathcal{U}_k$ for $2^k \leq i < m$. Thus, we can continue (6.17) to get,

$$w_{n'} r_k \left(\sum_{i=0}^{2^k - 1} \alpha_i w_i \right) = \mathbb{D}_k \left(w_{n'} r_k \left(\sum_{i=0}^{m} \alpha_i w_i \right) \right) = \mathbb{D}_k(w_n x). \tag{6.18}$$

Next, consider the the right summation on the right hand side of (6.16). Using (6.14), we find that

$$w_{n'} r_k \left(\sum_{i=2^k}^{n} \alpha_i w_i \right) = w_{n'} \left(\sum_{i=0}^{n'} \beta_i w_i \right) = w_{n'} \mathbb{P}_{n'} \left(\sum_{i=0}^{n'} \beta_i w_i \right),$$

for certain $\beta_i \in \mathbb{C}$, where in fact $\beta_i = \pm \alpha_{i+2^k}$, with the sign depending on n (the precise equality is irrelevant for the rest of the proof). Since $n' < n$, we continue this equation by induction. Taking into account the binary decomposition of

$n' = n - 2^k$ we find,

$$\begin{aligned} w_{n'} r_k \left(\sum_{i=2^k}^{n} \alpha_i w_i \right) =& \mathbb{E}_{-1} \left(w_{n'} \left(\sum_{i=0}^{n'} \beta_i w_i \right) \right) + \sum_{s \text{ with } \gamma_s = 1, s \neq k} \mathbb{D}_s \left(w_{n'} \left(\sum_{i=0}^{n'} \beta_i w_i \right) \right) \\ =& \mathbb{E}_{-1} \left(w_n \left(\sum_{i=2^k}^{n} \alpha_i w_i \right) \right) + \sum_{s \text{ with } \gamma_s = 1, s \neq k} \mathbb{D}_s \left(w_n \left(\sum_{i=2^k}^{n} \alpha_i w_i \right) \right). \end{aligned} \tag{6.19}$$

Now, note that $\mathbb{E}_{-1}(w_n w_i) \neq 0$ if and only if $i = n$. Furthermore, let $i > n$ and let $i = \sum_{s=0}^{\infty} \epsilon_s 2^s$, with $\epsilon_s \in \{0, 1\}$. Looking back at (6.13), we see that $w_n w_i \in \mathcal{U}_j$, where j is the largest number such that $\gamma_j \neq \epsilon_j$. Morever, since $i > n$ we have in fact $\gamma_j = 0$ and $\varepsilon_j = 1$. Hence, for $i > n$, we have $\sum_{s \text{ with } \gamma_s = 1, s \neq k} \mathbb{D}_s(w_n w_i) = 0$. Using these observations, we continue (6.19),

$$\begin{aligned} w_{n'} r_k \left(\sum_{i=2^k}^{n} \alpha_i w_i \right) =& \mathbb{E}_{-1} \left(w_n \left(\sum_{i=0}^{m} \alpha_i w_i \right) \right) + \sum_{s \text{ with } \gamma_s = 1, s \neq k} \mathbb{D}_s \left(w_n \left(\sum_{i=0}^{m} \alpha_i w_i \right) \right) \\ =& \mathbb{E}_{-1} \left(w_n x \right) + \sum_{s \text{ with } \gamma_s = 1, s \neq k} \mathbb{D}_s \left(w_n x \right). \end{aligned} \tag{6.20}$$

It is now clear that filling in (6.18) and (6.20) into (6.16) yields the induction hypotheses. ☐

Remark 6.3.5. In particular, it follows that for a fixed $n \in \mathbb{N}$ the map $\mathbb{P}_n$ has a unique extension to R_α which is both bounded and normal. We replace the notation $\mathbb{P}_n$ by its normal extension

$$\mathbb{P}_n : R_\alpha \to R_\alpha.$$

Note that we do not claim yet that the bound of $\mathbb{P}_n$ is uniform in n. In fact, this is true as we prove in the remainder of this section.

Recall from Remark 6.1.3 that left multiplication of an element $x \in R_\alpha$ on $L^p(R_\alpha)_{\text{left}}$ can be obtained by complex interpolation. We can also interpolate the maps $\mathbb{D}_s, \mathbb{E}_s, \mathbb{P}_s$ to get maps

$$\begin{aligned} \mathbb{D}_s^p &: \mathcal{L}^p(R_\alpha) \to \mathcal{L}^p(N_s), \\ \mathbb{E}_s^p &: \mathcal{L}^p(R_\alpha) \to \mathcal{L}^p(N_s), \\ \mathbb{P}_s^p &: \mathcal{L}^p(R_\alpha) \to \mathcal{L}^p(R_\alpha), \end{aligned}$$

where $1 \leq p \leq \infty$. Now, by functoriality of the complex interpolation method, we find the following corollary.

Corollary 6.3.6. *Let $1 \leq p \leq \infty$. For every $x \in L^p(R_\alpha)_{\mathrm{left}}, n \in \mathbb{N}$:*

$$w_n \mathbb{P}_n^p(x) = \mathbb{E}_{-1}^p(w_n x) + \sum_{s \text{ with } \gamma_s \neq 0} \mathbb{D}_s^p(w_n x), \tag{6.21}$$

where $\gamma_s \in \{0,1\}$ are such that $n = \sum_{s=0}^{\infty} \gamma_s 2^s$.

At this point it is usefull to recall the definition of a Schauder decomposition.

Definition 6.3.7 (Section 1.g of [63])**.** Let $\mathcal{X}$ be a Banach space and let $\mathbf{X} = (\mathcal{X}_s)_{s\in\mathbb{N}}$ be a sequence of closed subspaces of $\mathcal{X}$. Then, $\mathbf{X}$ is called a *Schauder decomposition* if every $x \in \mathcal{X}$ has a unique decomposition

$$x = \sum_{s=0}^{\infty} x_s, \qquad \text{where } \; x_s \in \mathcal{X}_s. \tag{6.22}$$

Lemma 6.3.8 (Section 1.g of [63])**.** *A sequence $\mathbf{X} = (\mathcal{X}_s)_{s\in\mathbb{N}}$ of closed subspaces of $\mathcal{X}$ is a Schauder decomposition if the linear span of $\cup_{s\in\mathbb{N}} \mathcal{X}_s$ is dense in $\mathcal{X}$ and furthermore, there is a constant C such that*

$$\| \sum_{s=0}^{n} x_s \|_{\mathcal{X}} \leq C \| \sum_{s=0}^{m} x_s \|_{\mathcal{X}},$$

for every $x_s \in \mathcal{X}_s$ and $n < m$.

We also need the notion of an unconditional Schauder basis. Let $\mathbf{X} = (\mathcal{X}_s)_{s\in\mathbb{N}}$ be a Schauder decomposition of $\mathcal{X}$. For $A \subseteq \mathbb{N}$, consider the projection:

$$\mathbb{T}_A : \mathcal{X} \to \mathcal{X} : x = \sum_{s=0}^{\infty} x_s \mapsto \sum_{s\in A} x_s,$$

where, of course, we mean that $x_s \in \mathcal{X}_s$.

Lemma 6.3.9 (Proposition 1.c.6 and its subsequent remarks in [63])**.** *The following are equivalent:*

1. *For every $A \subseteq \mathbb{N}$, the map $\mathbb{T}_A$ is bounded.*
2. *For every $x \in \mathcal{X}$ with $x = \sum_{s=0}^{\infty} x_s$, where $x_s \in \mathcal{X}_s$ and for every choice $\varepsilon_s \in \{-1, 1\}, s \in \mathbb{N}$, the sum $\sum_{s=0}^{\infty} \varepsilon_s x_s$, is convergent.*

Moreover, if these conditions are satisfied, then there is a constant C such that for every $A \subseteq \mathbb{N}$, we have $\|\mathbb{T}_A\| \leq C$.

If $(\mathcal{X}_s)_{s\in\mathbb{N}}$ satisfies the equivalent conditions of Lemma 6.3.9, then this sequence is called an *unconditional* Schauder decomposition.

Note that $(N_s)_{s\in\mathbb{N}}$ is an increasing filtration of von Neumann algebras such that its union is σ-weakly dense in R_α. Moreover, R_α is equipped with the faithful, normal state ρ_α. Therefore, Theorem 6.1.4 may be applied and we see that (2) of Lemma 6.3.9 holds for the decomposition $(\mathbb{D}_s^p(L^p(R_\alpha)_{\mathrm{left}}))_{s\in\mathbb{N}}$.

Proposition 6.3.10. *Let $1 < p < \infty$. Then, $(\mathbb{D}_s^p(L^p(R_\alpha)_{\rm left}))_{s\in\mathbb{N}}$ is an unconditional Schauder decomposition of $L^p(R_\alpha)_{\rm left}$.*

We are now in a position to prove the main theorem of this section.

Theorem 6.3.11. *For $1 < p < \infty$, the Walsh system $\mathbf{w}$ forms a Schauder basis in $L^p(R_\alpha)_{\rm left}$.*

Proof. It follows from Proposition 6.2.2 that the linear span of the Walsh system is dense in $L^p(R_\alpha)_{\rm left}$. We have to prove that (6.11) with $\mathcal{X} = L^p(R_\alpha)_{\rm left}$ holds for a certain C. Equivalently, we must prove that the projections $\mathbb{P}_n^p$ are uniformly bounded in n. Recall that by Theorem 6.3.4 for $x \in L^p(R_\alpha)_{\rm left}, n \in \mathbb{N}$:

$$\mathbb{P}_n^p(x) = w_n \mathbb{E}_{-1}^p(w_n x) + w_n \sum_{s \text{ with } \gamma_s \neq 0} \mathbb{D}_s^p(w_n x), \tag{6.23}$$

where $\gamma_s \in \{0, 1\}$ are such that $n = \sum_{s=0}^\infty \gamma_s 2^s$. Now, left multiplication with w_n is an isometric map on $L^p(R_\alpha)_{\rm left}$. Hence,

$$\|\mathbb{P}_n^p\| = \|\mathbb{E}_{-1}^p + \sum_{s \text{ with } \gamma_s \neq 0} \mathbb{D}_s^p\| \leq \|\mathbb{E}_{-1}^p\| + \|\sum_{s \text{ with } \gamma_s \neq 0} \mathbb{D}_s^p\|, \tag{6.24}$$

Since we assume that $1 < p < \infty$, the decomposition $(\mathbb{D}_s^p(L^p(R_\alpha)_{\rm left}))_{s\in\mathbb{N}}$ is unconditional. Hence, it follows from Lemma 6.3.9 that the right hand side of (6.24) is uniformly bounded in n. □

Remark 6.3.12. We would like to emphasize that the fact that left multiplication is compatible with the left injection forms an essential step in the proof of Theorem 6.3.11. If one considers L^p-spaces with respect to the right injection, one can prove that for $1 \leq p \leq \infty, x \in L^p(R_\alpha)_{\rm right}$ and $n \in \mathbb{N}$:

$$\mathbb{P}_n^{p,\sharp}(x) w_n = \mathbb{E}_{-1}^{p,\sharp}(x w_n) + \sum_{s \text{ with } \gamma_s \neq 0} \mathbb{D}_s^{p,\sharp}(x w_n), \tag{6.25}$$

where $\gamma_s \in \{0, 1\}$ are such that $n = \sum_{s=0}^\infty \gamma_s 2^s$. Here, the maps $\mathbb{D}_s^{p,\sharp}, \mathbb{E}_s^{p,\sharp}, \mathbb{P}_s^{p,\sharp}$ are the interpolated maps of $\mathbb{D}_s, \mathbb{E}_s, \mathbb{P}_s$ with respect to the right injection. Completely analogously, one can now prove that $\mathbf{w}$ forms a Schauder basis in the right L^p-spaces.

6.4 The Walsh basis in the hyperfinite factor of type III$_1$

Here, we construct a Walsh basis in the L^p-spaces associated with the hyperfinite factor of type III$_1$. The construction follows the line of [70, Section 7], however the arguments are different as they rely on Section 6.3.

Consider arbitrary von Neumann algebras N and M with with faithful, normal states ϕ and ψ. For the modular automorphism group of $\phi\otimes\psi$, we have

$$\sigma_t^{\phi\otimes\psi} = \sigma_t^{\phi}\otimes\sigma_t^{\psi}, \qquad t\in\mathbb{R}.$$

Therefore, $N\otimes 1$ is a von Neumann subalgebra of $N\otimes M$ that is globally invariant under $\sigma^{\phi\otimes\psi}$. There exists a $\phi\otimes\psi$-preserving conditional expectation value $\mathbb{E}_N : N\otimes M\to N\otimes 1$, [75, Theorem IX.4.2]. Suppose that $\mathbf{v} = (v_j)_{j\in\mathbb{N}}$ is a sequence in M with $v_j^* v_j$ equal to a multiple of the identity. We define maps:

$$\mathbb{F}_{N,j}(x) = (1\otimes v_j)\mathbb{E}_N((1\otimes v_j^*)x), \qquad x\in N\otimes M.$$

Since $\mathbb{F}_{N,j}$ is the composition of left multiplications and $\mathbb{E}_N$, we can use the complex interpolation method to get a bounded map:

$$\mathbb{F}^p_{N,j} : \mathcal{L}^p(N\otimes M)\to\mathcal{L}^p(N\otimes 1) = \mathcal{L}^p(N). \tag{6.26}$$

Similarly, we can consider a $\phi\otimes\psi$-preserving conditional expectation value $\mathbb{E}_M : N\otimes M\to 1\otimes M$. If $\mathbf{u} = (u_i)_{i\in\mathbb{N}}$ is a sequence in N with $u_i^* u_i$ equal to a multiple of the identity, then we set:

$$\mathbb{F}_{M,i}(x) = (u_i\otimes 1)\mathbb{E}_M((u_i^*\otimes 1)x), \qquad x\in N\otimes M.$$

Interpolating this map, yields a map $\mathbb{F}^p_{M,i} : \mathcal{L}^p(N\otimes M)\to\mathcal{L}^p(1\otimes M) = \mathcal{L}^p(M)$.

The following theorem can be proved similarly as [70, Theorem 7.1]. For completeness and convenience of the reader, we give the proof. Recall that the shell enumeration is an enumeration of $\mathbb{N}\times\mathbb{N}$, which assigns to a pair (i,j) the number

$$\varphi(i,j) = \begin{cases} j^2+i & \text{if } i\leq j,\\ (i+1)^2-j-1 & \text{if } i>j.\end{cases}$$

Theorem 6.4.1. *Let $1\leq p\leq\infty$. Suppose that $\mathbf{u} = (u_i)_{i\in\mathbb{N}}$ and $\mathbf{v} = (v_j)_{j\in\mathbb{N}}$ are sequences of linearly independent unitaries in N and respectively M. Denote the corresponding projections by $\mathbb{F}^p_{N,j}$ and $\mathbb{F}^p_{M,i}$ and suppose that $(\mathbb{F}^p_{N,j}(L^p(N\otimes M)_{\mathrm{left}})_{j\in\mathbb{N}}$ and $(\mathbb{F}^p_{M,i}(L^p(N\otimes M)_{\mathrm{left}}))_{i\in\mathbb{N}}$ are Schauder decompositions of $N\otimes M$. Then, $\mathbf{u}\otimes\mathbf{v} = (u_i\otimes v_j)_{i,j\in\mathbb{N}}$ taken in the shell enumeration is a Schauder basis for $L^p(N\otimes M)_{\mathrm{left}}$.*

Proof. Let $\mathbf{z} = \mathbf{u}\otimes\mathbf{v}$ and write $\mathbf{z} = (z_k)_{k\in\mathbb{N}}$. Let $n,m\in\mathbb{N}$ be such that $n<m$ and consider the sum $\sum_{i=0}^m \alpha_i z_i$, where $\alpha_i\in\mathbb{C}$. Let $l\in\mathbb{N}$ be such that $l^2\leq n<(l+1)^2$. There are two cases: either $l^2\leq n\leq l^2+l$ or $l^2+l<n<(l+1)^2$. We treat the first case, since the second case can be handled similarly. First, we compute:

$$\begin{aligned}\|\sum_{k=0}^{n}\alpha_k z_k\|_p &\leq \|\sum_{k=0}^{l^2-1}\alpha_k z_k\|_p + \|\sum_{k=l^2}^{n}\alpha_k z_k\|_p\\ =&\|\sum_{0\leq i,j<l}\alpha_{\varphi(i,j)} u_i\otimes v_j\|_p + \|\sum_{i=0}^{n-l^2}\alpha_{\varphi(i,l)} u_i\otimes v_l\|_p\end{aligned}$$

For the two terms on the right hand side, we find:

$$\| \sum_{0 \leq i,j < l} \alpha_{\varphi(i,j)} u_i \otimes v_j \|_p = \| \mathbb{P}^p_{M,l-1} \mathbb{P}^p_{N,l-1} \left(\sum_{k=0}^m \alpha_k z_k \right) \|_p,$$

$$\| \sum_{i=0}^{n-l^2} \alpha_{\varphi(i,l)} u_i \otimes v_l \|_p = \| \mathbb{F}^p_{M,l} \mathbb{P}^p_{N,n-l^2} \left(\sum_{k=0}^m \alpha_k z_k \right) \|_p,$$

where $\mathbb{P}^p_{N,s} = \sum_{i=0}^s \mathbb{F}^p_{N,i}$ and $\mathbb{P}^p_{M,s} = \sum_{j=0}^s \mathbb{F}^p_{M,j}$. Since we assume that the sequences $(\mathbb{F}^p_{N,j}(L^p(N \otimes M)_{\mathrm{left}}))_{j \in \mathbb{N}}$ and $(\mathbb{F}^p_{M,i}(N \otimes M)_{\mathrm{left}}))_{i \in \mathbb{N}}$ are Schauder decompositions of $L^p(N \otimes M)_{\mathrm{left}}$, the projections $\mathbb{P}^p_{N,s}$ and $\mathbb{P}^p_{M,s}$ are uniformly bounded in s, c.f. Lemma 6.3.8. It follows that there is a constant C such that relation (6.11) is holds. □

Choose $0 < \alpha, \alpha' < \frac{1}{2}$ such that R_α and $R_{\alpha'}$ are factors of type III$_\lambda$ and III$_{\lambda'}$ with $\frac{\log \lambda}{\log \lambda'} \notin \mathbb{Q}$ and $\lambda = \frac{\alpha}{1-\alpha}, \lambda' = \frac{\alpha'}{1-\alpha'}$. In that case, the tensor product $R_\alpha \otimes R_{\alpha'}$ is isomorphic to the hyperfinite factor of type III$_1$, see [14], [36]. Consider the Walsh basis $\mathbf{w}$ in $L^p(R_\alpha)_{\mathrm{left}}$ and let $\mathbf{w}'$ be the Walsh basis in $L^p(R_{\alpha'})_{\mathrm{left}}$. Let $\mathbb{F}^p_{\alpha,j} (= \mathbb{F}^p_{R_\alpha,j})$ be the projection constructed in (6.26) and similarly consider $\mathbb{F}^p_{\alpha',i} (= \mathbb{F}^p_{R_{\alpha'},j})$.

Proposition 6.4.2. *Let $1 < p < \infty$. The decomposition $(\mathbb{F}^p_{\alpha,j}(L^p(R_\alpha \otimes R_{\alpha'})_{\mathrm{left}}))_{j \in \mathbb{N}}$ is a Schauder decomposition of $L^p(R_\alpha \otimes R_{\alpha'})_{\mathrm{left}}$. Similarly, the sequence given by $(\mathbb{F}^p_{\alpha',j}(L^p(R_\alpha \otimes R_{\alpha'})_{\mathrm{left}}))_{j \in \mathbb{N}}$ is a Schauder decomposition of $L^p(R_\alpha \otimes R_{\alpha'})_{\mathrm{left}}$.*

Proof. We only prove the first statement, since the second one can be proved similarly. Set $\mathbb{P}_{\alpha,n} = \sum_{j=0}^n \mathbb{F}_{\alpha,j}$ and $\mathbb{P}^p_{\alpha,n} = \sum_{j=0}^n \mathbb{F}^p_{\alpha,j}$. In view of Lemma 6.3.8, we must prove that $\mathbb{P}^p_{\alpha,n}$ is uniformly bounded in n.

Let $m > n$. Consider an element $x = \sum_{0 \leq i,j \leq m} \alpha_{i,j} w_i \otimes w'_j$ with $\alpha_{i,j} \in \mathbb{C}$. We find

$$\mathbb{P}_{\alpha,n} \left(\sum_{0 \leq i,j \leq m} \alpha_{i,j} w_i \otimes w'_j \right) = \sum_{0 \leq i \leq m,\, 0 \leq j \leq n} \alpha_{i,j} w_i \otimes w'_j$$

$$= (\iota \otimes \mathbb{P}_n) \left(\sum_{0 \leq i,j \leq m} \alpha_{i,j} w_i \otimes w'_j \right)$$

In particular, the normality of $\mathbb{P}_{\alpha,n}$ implies that $\mathbb{P}_{\alpha,n} = (\iota \otimes \mathbb{P}_n)$, where ι is the identity on R_α.

Note that $R_\alpha \otimes N_s$ is a von Neumann subalgebra of $R_\alpha \otimes R_{\alpha'}$ that is globally invariant under the modular automorphism group of $\rho_\alpha \otimes \rho_{\alpha'}$. Let $\mathbb{E}_{\alpha,s} : R_\alpha \otimes R_{\alpha'} \to R_\alpha \otimes N_s$ be the associated $\rho_\alpha \otimes \rho_{\alpha'}$-preserving conditional expectation value. Consider also the $\rho_{\alpha'}$-preserving conditional expection value $\mathbb{E}_s : R_{\alpha'} \to$

N_s. Clearly, the uniqueness of $(\rho_\alpha \otimes \rho_{\alpha'})$-preserving conditional expectations implies that:

$$\mathbb{E}_{\alpha,s} = \iota \otimes \mathbb{E}_s.$$

Recall that we defined $\mathbb{D}_s = \mathbb{E}_s - \mathbb{E}_{s-1}$. Similarly, set $\mathbb{D}_{\alpha,s} = \mathbb{E}_{\alpha,s} - \mathbb{E}_{\alpha,s-1}$.

Now, we obtain the following equalities from Theorem 6.3.4.

$$\begin{aligned}
(1 \otimes w_n')\mathbb{P}_{\alpha,n}(x) &= (\iota \otimes w_n')(\iota \otimes \mathbb{P}_n)(x) \\
&= \left(\iota \otimes \left(\mathbb{E}_{-1} + \sum_{i \text{ with } \gamma_i = 1} \mathbb{D}_i\right)\right)((1 \otimes w_n')x) \\
&= \left(\mathbb{E}_{\alpha,-1} + \sum_{i \text{ with } \gamma_i = 1} \mathbb{D}_{\alpha,i}\right)((1 \otimes w_n')x),
\end{aligned}$$

where $n = \sum_{i=0}^{\infty} \gamma_i 2^i$ with $\gamma_i \in \{0,1\}$. Interpolating this equation, and observing that left multiplication with $(1 \otimes w_n')$ is an isometric map on $\mathcal{L}^p(R_\alpha \otimes R_{\alpha'})$, we find that:

$$\|\mathbb{P}^p_{\alpha,n}\| = \|\mathbb{E}^p_{\alpha,-1} + \sum_{i \text{ with } \gamma_i = 1} \mathbb{D}^p_{\alpha,i}\| \leq \|\mathbb{E}^p_{\alpha,-1}\| + \|\sum_{i \text{ with } \gamma_i = 1} \mathbb{D}^p_{\alpha,i}\|. \tag{6.27}$$

By remarks similar to the ones preceeding Proposition 6.3.10, it follows from Theorem 6.1.4 that the decomposition $(\mathbb{D}^p_{\alpha,i}(L^p(R_\alpha \otimes R_{\alpha'})_{\text{left}})_{i\in\mathbb{N}}$ is an unconditional Schauder decomposition of $L^p(R_\alpha \otimes R_{\alpha'})_{\text{left}}$. Hence, Lemma 6.3.9 implies that the right hand side of (6.27) is uniformly bounded in n. $\square$

Proposition 6.4.2 implies that we may apply Theorem 6.4.1.

Theorem 6.4.3. *Let $1 < p < \infty$. The Walsh system $\mathbf{w} \otimes \mathbf{w}' = (w_i \otimes w_j')_{i,j\in\mathbb{N}}$ taken in the shell enumeration is a Schauder basis in $L^p(R_\alpha \otimes R_{\alpha'})_{\text{left}}$; the L^p-space associated with the hyperfinite III$_1$ factor.*

Remark 6.4.4. In general a tensor product of two L^p-spaces, each with unconditional decomposition, does not produce a L^p-space where the tensor product of the given decompositions is unconditional. The simplest example is a couple of Schatten classes with row and column decompositions.

6.5 Classical L^p-spaces

For $s \in \mathbb{N}$, consider the diagonal subalgebra $A_s \subseteq N_s$. The weak closure of $\cup_{s\in\mathbb{N}} A_s$ in R_α forms an abelian von Neumann algebra A_α, which is isomorphic to $L^\infty([0,1], \mu_\alpha)$. Here, μ_α is the measure determined by:

$$\mu_\alpha\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) = \prod_{i=0}^{n-1} \left[(1-\gamma_i)\alpha + \gamma_i(1-\alpha)\right],$$

where $0 \leq k < 2^n$ and $\gamma_i \in \{0,1\}$ are such that $k = \sum_{i=0}^{n-1} \gamma_i 2^i$, see [42, Section 12.3]. In particular, $A_{\frac{1}{2}}$ is isomorphic to $L^\infty([0,1],\mu)$, where μ is the Lebesgue measure.

The modular automorphism group σ leaves $\cup_{s\in\mathbb{N}} A_s$ and hence A_α invariant. From Section 6.1, it follows that $L^p(A_\alpha)_{\mathrm{left}}$ is a closed subspace of $L^p(R_\alpha)_{\mathrm{left}}$. Moreover, there exists a conditional expectation value $\mathbb{E}_{A_\alpha} : R_\alpha \to A_\alpha$. Since $\mathbb{E}_{A_\alpha}$ projects on the diagonal matrices, we find that it acts on the Walsh system $\mathbf{w}$ by:

$$\mathbb{E}_{A_\alpha}(w_n) = \begin{cases} w_n & \text{if } n = \sum_{i=0}^\infty \gamma_i 2^i \text{ with } \gamma_{2i+1} = 0 \text{ for every } i, \\ 0 & \text{else.} \end{cases}$$

Indeed, it follows from (6.13) that w_n is diagonal if and only if the odd digits in the binary decomposition of n vanish. Let $\mathbf{z}$ be the subsequence of $\mathbf{w}$ of vectors in the range of the projection $\mathbb{E}_{A_\alpha}$. Clearly, it follows from Theorem 6.3.11 that $\mathbf{z}$ forms a Schauder basis in $L^p(A_\alpha)_{\mathrm{left}}$ for $1 < p < \infty$. Explicitly, this system is constructed as follows. Recall that we defined the Rademacher matrices in (6.12). Set:

$$z_n = \bigotimes_{i=0}^\infty r^{(\gamma_i,0)}, \qquad n = \sum_{i=0}^\infty \gamma_i 2^i, \ \ \gamma_i \in \{0,1\}.$$

Then, $\mathbf{z} = (z_n)_{n\in\mathbb{N}}$.

Corollary 6.5.1. *Let $1 < p < \infty$. The system $\mathbf{z}$ forms a Schauder basis in $L^p(A_\alpha)_{\mathrm{left}}$. Under the isomorphism $L^p(A_\alpha)_{\mathrm{left}} \simeq L^p([0,1],\mu_\alpha)$, we obtain the classical Walsh system (6.1).*

Appendix A

Here, we recall the more technical material used throughout this thesis. The material includes direct integration, some results on C*-algebraic and von Neumann algebraic weights. We prove as well some approximation lemmas and density results.

A.1 Direct integration

The theory of direct integration was first developed by Dixmier, see his book [21]. We follow his book, but introduce the necessary structures directly from fundamental sequences. This is for two reasons. First of all, this keeps things short. Though the reader interested in the general theory should definitely read [21]. Secondly, most of the additional technical results we encounter in this thesis have easier proofs in terms of fundamental sequences. The definitions we give here are equivalent to the ones by Dixmier. We put appropriate references to Dixmier's book to indicate where a proof of this equivalence can be found. For direct integration of unbounded operators we refer to [61] and [66], where we mainly follow [61].

Finally, we remark that a subset A of a Hilbert space $\mathcal{H}$ is called *total* if the closure of the linear span of A equals $\mathcal{H}$.

Notation A.1.1. In this section, let (X, ν) be a *standard measure space*, i.e. there is a neglectable subset $Y \subseteq X$, such that $X\backslash Y$ carries the Borel σ-algebra of a complete separable metric space. When we encounter definitions as measurable, neglectable, almost everywhere, et cetera, this is with respect to ν. All fields of vectors, Hilbert spaces, et cetera, we encounter are fields over X.

Definition A.1.2 (Proposition II.1.4 of [21])**.** Let $(\mathcal{H}_x)_x$ be a field of Hilbert spaces. Suppose that for every $n \in \mathbb{N}$, there exists a field of vectors $(e_x^n)_x$ with $e_x^n \in \mathcal{H}_x$ such that:

(1) For every $n, m \in \mathbb{N}$, we have that $x \mapsto \langle e_x^n, e_x^m \rangle$ is a measurable function;

(2) For every $x \in X$, the set $\{e_x^n \mid n \in \mathbb{N}\}$ is total in $\mathcal{H}_x$.

The sequence $(e_x^n)_x$ is called a *fundamental sequence*. If such a fundamental sequence exists, the field $(\mathcal{H}_x)_x$ is called *measurable*.

Let $(\mathcal{H}_x)_x$ be a field of Hilbert spaces. Suppose that both $(e_x^n)_x$ and $(f_x^n)_x$ are both fundamental sequences. It is not true in general that for every $n, m \in \mathbb{N}$, the function $x \mapsto \langle e_x^n, f_x^m \rangle$ is measurable. So how a field of Hilbert spaces turns into a measurable field depends on the fundamental sequence. We will normally keep the measurable structure out of our terminology and notation as it is the most common notation in the literature. However, the reader has to keep in mind that there is always a measurable sequence in the background.

From now on, we fix a measurable field of Hilbert spaces $(\mathcal{H}_x)_x$ with fundamental sequence $(e_x^n)_x$, $n \in \mathbb{N}$.

Proposition A.1.3 (Proposition II.1.1 (i) of [21]). *For every $n \in \mathbb{N} \cup \{\infty\}$, the set $X_n = \{x \in X \mid \dim(\mathcal{H}_x) = n\}$ is measurable.*

Definition A.1.4 (Proposition II.1.2 of [21]). A field of vectors $(v_x)_x$ with $v_x \in \mathcal{H}_x$ is called *measurable* if $x \mapsto \langle v_x, e_x^n \rangle$ is a measurable function for all $n \in \mathbb{N}$.

Definition A.1.5. A field of vectors $(v_x)_x$ with $v_x \in \mathcal{H}_x$ is called *essentially bounded* if $x \mapsto \|v_x\|$ is essentially bounded.

In Dixmier's book [21], the following lemma is true by definition. The equivalence with our definition follows from the following reference.

Lemma A.1.6 (Proposition II.1.2 of [21]). *If $(v_x)_x, (w_x)_x$ are measurable fields of vectors, then $x \mapsto \langle v_x, w_x \rangle$ is measurable.*

It is proved in [21] that the following construction indeed defines a Hilbert space.

Definition A.1.7 (Proposition II.1.5 of [21]). Consider the linear space $\mathcal{R}$ of measurable fields of vectors $(v_x)_x$ for which $x \mapsto \langle v_x, v_x \rangle$ is integrable. Such a sequence is called *square integrable*. Define a (degenerate) inner product on $\mathcal{R}$ by

$$\langle (v_x)_x, (w_x)_x \rangle = \int \langle v_x, w_x \rangle d\nu(x).$$

We define the *direct integral Hilbert space* $\int_X^{\oplus} \mathcal{H}_x d\nu(x) = \int^{\oplus} \mathcal{H}_x d\nu(x)$ as the quotient of $\mathcal{R}$ by the nilspace with respect to this inner product.

We also write $\int_X^{\oplus} v_x d\nu(x) = \int^{\oplus} v_x d\nu(x)$ for $(v_x)_x$ if we particularly want to see $(v_x)_x$ as an element of $\int^{\oplus} \mathcal{H}_x d\nu(x)$.

We mention the following examples.

Example A.1.8. If X is a locally compact, second countable group G and ν is the left Haar measure, we get $\int_G^{\oplus} \mathbb{C} d\nu(x) = L^2(G)$.

Example A.1.9. If ν has only discrete mass points, i.e. for every $x \in X$ we have $\nu(\{x\}) > 0$, then $\int_X^{\oplus} \mathcal{H}_x d\nu(x) = \oplus_{x\in X}\mathcal{H}_x$. In particular, one is able to obtain the quantum Peter-Weyl Theorem 1.5.8 from the quantum Plancherel Theorem 1.6.1.

We need the following fact about tensor product spaces, which from our definition becomes trivial.

Proposition A.1.10 (Proposition II.1.10). *Let $(\mathcal{H}_x)_x$ and $(\mathcal{K}_x)_x$ be measurable fields of Hilbert spaces with respective fundamental sequences $(e_x^n)_x$ and $(f_x^n)_x$, where $n \in \mathbb{N}$. Then, $(\mathcal{H}_x \otimes \mathcal{K}_x)_x$ is a measurable field of Hilbert spaces for the fundamental sequence $(e_x^n \otimes f_x^m)_x, n, m \in \mathbb{N}$.*

We always equip a direct integral of tensor products implicitly with the fundamental sequence described in Proposition A.1.10.

We introduce direct integrals of closed unbounded operators. In particular, this defines direct integrals of bounded operators.

Definition A.1.11 (p. 15 (1) - (3) of [61]). Let $(A_x)_x$ be a field of densely defined, closed (possibly unbounded) operators A_x acting on the Hilbert space $\mathcal{H}_x$. Then, the field $(A_x)_x$ is called *measurable* if there is a sequence of measurable fields of vectors $(f_x^n)_x, n \in \mathbb{N}$ such that:

(1) $f_x^n \in \mathrm{Dom}(A_x)$ for all $x \in X$;

(2) $A_x f_x^n$ is measurable for all $x \in X$;

(3) For every $x \in X$, the linear span of f_x^n is a core for A_x. Equivalently, for every $x \in X$, the set $\{(f_x^n, A_x f_x^n) \mid n \in \mathbb{N}\}$ is total in the graph of A_x with respect to the graph norm.

In that case, $(f_x^n)_x$ is called a *fundamental sequence* for $(A_x)_x$.

Note that we may replace $(e_x^n)_x$ by $(f_x^n)_x$ if necessary so that the fundamental sequences for the field of Hilbert spaces equals the one for the field of operators. In case $(A_x)_x$ is a measurable field of densely defined, closed operators, we define the operator $\int_X^{\oplus} A_x d\nu(x) = \int^{\oplus} A_x d\nu(x)$ with domain the square integrable fields of vectors $(v_x)_x$ for which $(A_x v_x)_x$ is square integrable, by means of the following formula:

$$\int^{\oplus} A_x d\nu(x) \int^{\oplus} v_x d\nu(x) = \int^{\oplus} A_x v_x d\nu(x).$$

Now, it turns out that the direct integral operator is automatically densely defined and closed.

Proposition A.1.12 (Theorem 1.4 of [61]). *Let $(A_x)_x$ be a measurable field of densely defined, closed operators. Then, $\int^\oplus A_x d\nu(x)$ is closed and densely defined. Moreover, $(\int^\oplus A_x d\nu(x))^* = \int^\oplus A_x^* d\nu(x)$. In particular, if A_x is self-adjoint for almost all x, then $\int^\oplus A_x d\nu(x)$ is self-adjoint.*

Theorem A.1.13 (Theorem 1.10 of [61]). *Let $(A_x)_x$ be a measurable field of self-adjoint operators. Let $f : \mathbb{R} \to \mathbb{C}$ be a (Borel) measurable function. Then, the functional calculus commutes with direct integration, i.e.*

$$f\left(\int^\oplus A_x d\nu(x)\right) = \int^\oplus f(A_x) d\nu(x).$$

Definition A.1.14. Let f be a measurable, essentially bounded function on X, then the canonical multiplication operator $\int^\oplus f(x) 1_{\mathcal{H}_x} d\nu(x)$ is called a *diagonalizable operator*.

A.2 Tomita-Takesaki theory

We recall the basics of Tomita-Takesaki theory here. We let M be a von Neumann algebra and φ a normal semi-finite faithful weight. We set $\mathfrak{n}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$. It is well-known that $\mathfrak{n}_\varphi$ is a left ideal. We constructe a (cyclic) GNS-representation as follows. Define an inner product

$$\mathfrak{n}_\varphi \times \mathfrak{n}_\varphi \to \mathbb{C} : (x, y) \mapsto \varphi(y^* x).$$

Let $\mathcal{H}$ denote the compltion of $\mathfrak{n}_\varphi$ with respect to this inner product. We denote $\Lambda : \mathfrak{n}_\varphi \to \mathcal{H}$ for the canonical embedding. It is well-known that Λ is σ-weakly/weakly continuous and therefore also σ-strong-$*$/norm continuous, the argument is similar to the one in the proof of Lemma A.6.1. Now, we have a representation π of M on $B(\mathcal{H})$ given by

$$\pi(x)\Lambda(y) = \Lambda(xy), \qquad x \in M, y \in \mathfrak{n}_\varphi.$$

From the faithfulness and normality of φ it follows that π is faithful and normal. Therefore, we will identify M with $\pi(M)$ if this does not lead to any confusion. Consider the anti-linear operator

$$T_0 : \Lambda(x) \mapsto \Lambda(x^*), \qquad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

Then, T_0 is preclosed and we denote its closure by T. Its polar decomposition is denoted by $T = J\nabla^{\frac{1}{2}}$, where J is called the *modular conjugation* and ∇ is called the *modular operator*. By Tomita-Takesaki theory:

$$\nabla^{it} M \nabla^{-it} = M, \text{ and } JMJ = M' \qquad t \in \mathbb{R}.$$

We define the modular automorphism group of the weight φ as σ^φ (in this thesis commonly denoted by σ) as a strongly continuous 1-parameter group of automorphisms of M defined as

$$\sigma_t(x) = \nabla^{it} x \nabla^{-it}.$$

Let $\mathcal{M}_\varphi$ denote the set of $x \in M$ that are analytic with respect to σ. This means that $\mathbb{R} \ni t \mapsto \sigma_t^\varphi(x)$ extends to an analytic map on $\mathbb{C}$. Recall that analyticity of a von Neumann algebra-valued function equals σ-weak analyticity.

Proposition A.2.1 (Result 1.2 of [58]). *Let $U \subseteq \mathbb{C}$ be open and consider a function $f : U \to M$. The function f is analytic if and only if it is σ-weakly analytic. That is, for every $z_0 \in U$,*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{A.1}$$

exists with respect to the norm, if and only if the limit (A.1) exists in the σ-weak topology.

The following fundamental property of the modular automorphsim group is used many times. It is a formulation, which is close to the KMS property of a weight [75]. However, this property suffices for our purposes.

Theorem A.2.2. *Let $x \in \mathfrak{m}_\varphi$ and $a \in \mathcal{M}_\varphi$. Then, $ax \in \mathfrak{m}_\varphi$ and $xa \in \mathfrak{m}_\varphi$. Moreover,*

$$\varphi(ax) = \varphi(x\sigma_{-i}(a)).$$

In particular, this relation holds if moreover $a \in \mathcal{T}_\varphi$.

A.3 C*-algebraic weights

Let A be a C*-algebra and let ϕ be a C*-algebraic weight on A. We denote again $\mathfrak{n}_\phi = \{x \in A \mid \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\phi = \mathfrak{n}_\phi^* \mathfrak{n}_\phi$. ϕ is called *densely defined* if $\mathfrak{m}_\phi$ is dense in A with respect to the norm. ϕ is called *lower semi-continuous* if for every $\lambda \in \mathbb{R}^+$, the set

$$\left\{x \in A^+ \mid \varphi(x) \leq \lambda\right\}$$

is closed. By standard techniques, one can define a GNS-representation for ϕ, which again is a triple $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$. In fact, we have the following definition.

Definition A.3.1. Let A be a C*-algebra with weight ϕ. A GNS-construction for ϕ is by definition a triple $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ such that:

1. $\mathcal{H}_\phi$ is a Hilbert space.
2. Λ_ϕ is a linear map from $\mathfrak{n}_\phi$ to $\mathcal{H}_\phi$ such that $\Lambda_\phi(\mathfrak{n}_\phi)$ is dense in $\mathcal{H}_\phi$ and for every $x, y \in \mathfrak{n}_\phi$ we have $\langle \Lambda_\phi(x), \Lambda_\phi(y)\rangle = \phi(y^*x)$.

3. π_ϕ is a representation of A on $\mathcal{H}_\phi$ such that $\pi_\phi(x)\Lambda_\phi(y) = \Lambda_\phi(xy)$ for every $x \in A$ and $y \in \mathfrak{n}_\phi$.

Every GNS-representation is unique up to a unitary transformation.

Next, we recall Kustermans' technique of lifting weights to the von Neumann algebra generated by the GNS-representation of A. Put $M = \pi_\phi(A)''$. It is described in [59, Section 1.7] how the weight ϕ can be lifted to a von Neumann algebraic weight $\tilde{\phi}$ on M. The lift can be characterized in the following way. We mention that this is not the original definition, but it is the one most suitable for our purposes.

Definition A.3.2 (Proposition 1.32 of [59]). Let A be a C*-algebra with densely defined, lower semi-continuous, non-zero weight ϕ with GNS-construction given by the triple $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$. Let $\Lambda_{\tilde{\phi}}$ be the σ-strong-$*$/norm closure of the mapping $\pi_\phi(\mathfrak{n}_\phi) \mapsto \mathcal{H}_\phi : \pi_\phi(x) \mapsto \Lambda_\phi(x)$. Define $\tilde{\phi}$ as the unique normal, semi-finite weight on $\pi_\phi(A)''$ such that $(\mathcal{H}_\phi, \iota, \Lambda_{\tilde{\phi}})$ is a GNS-construction for $\tilde{\phi}$.

The weight $\tilde{\phi}$ is called the *W*-lift* of ϕ. Faithfulness of ϕ does not automatically imply faithfulness of $\tilde{\phi}$. In fact, we use the following definition. Again we give a definition equivalent to the original one.

Definition A.3.3 (Proposition 1.35 of [59]). A densely defined, lower semi-continuous, non-zero C*-algebraic weight ϕ is called *approximate(ly) KMS* if the W*-lift $\tilde{\phi}$ is faithful.

We mention that KMS-weights are approximately KMS, so that the terminology is consistent with the definitions. Note that if $\tilde{\phi}$ is faithful, then $\tilde{\phi}$ satisfies the KMS-property as this is a consequence of Tomita-Takesaki theory, see [75].

A.4 The Radon-Nikodym derivative

In this section, we recall Connes' cocycle derivative theorem as well as Vaes' Radon-Nikodym derivative for von Neumann algebras.

Theorem A.4.1 (Theorem 3.3 of [75]). *Let M be a von Neumann algebra with normal, semi-finite, faithful weights φ and ψ. Let σ^φ and σ^ψ be their modular automorphism groups. There exists a unique σ-strongly continuous one parameter family $\{u_t\}$ of unitaries in M with the following properties:*

1. $u_{t+s} = u_s \sigma^\psi_s(u_t), s, t \in \mathbb{R}$.

2. $u_s \sigma^\psi_s(\mathfrak{n}^*_\varphi \cap \mathfrak{n}_\psi) = \mathfrak{n}^*_\varphi \cap \mathfrak{n}_\psi$.

3. *For each $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}^*_\psi$ and $y \in \mathfrak{n}^*_\varphi \cap \mathfrak{n}_\psi$, there exists a complex valued function F on the strip $\{z \in \mathbb{C} \mid 0 \leq \Im(z) \leq 1\}$ that is continuous on the whole strip and analytic on the interior of the strip, such that:*

$$F(t) = \varphi(u_t \sigma^\psi_t(y)x), \qquad F(t+i) = \psi(x u_t \sigma^\psi_t(y)).$$

4. $\sigma_t^{\varphi}(x) = u_t \sigma_t^{\psi}(x) u_t^*$, *where* $x \in M, t \in \mathbb{R}$.

We use the standard notation

$$\left(\frac{D\varphi}{D\psi}\right)_t = u_t.$$

Next, we recall the Radon-Nikodym theorem as proved by Vaes in [82], building on the work of Pedersen and Takesaki [68]. To this end, let M be a von Neumann algebra, together with normal, semi-finite, faithful weight φ. Let σ^{φ} be its modular automorphism group and let $(\mathcal{H}, \pi, \Lambda)$ be its GNS-representation. Let Q be a positive, self-adjoint operator affiliated with M that satisfies the relation $\sigma_t^{\varphi}(Q^{is}) = \lambda^{ist} Q^{is}$ for a certain constant $\lambda \in \mathbb{R}^+$. Consider the set:

$$\mathfrak{n}_0 = \{x \in M \mid xQ^{\frac{1}{2}} \text{ is bounded and } x \cdot Q^{\frac{1}{2}} \in \mathfrak{n}_{\varphi}\}.$$

Now, construct a map $\Gamma_0 : \mathfrak{n}_0 \to \mathcal{H}$ by setting $\Gamma_0(x) = \Lambda(x \cdot Q^{\frac{1}{2}})$. The map Γ_0 is σ-strongly-$*$/norm closable and we let Γ be its closure [82]. It is proved in [82] that there is a uniqe weight φ_Q that has GNS-representation $(\mathcal{H}, \pi, \Gamma)$. Formally it is given by $\varphi_Q(\cdot) = \varphi(Q^{\frac{1}{2}} \cdot Q^{\frac{1}{2}})$. Moreover, it is characterized by the following theorem.

Theorem A.4.2 (Theorem 5.5 of [82]). *Let M be a von Neumann algebra with normal, semi-finite, faithful weights φ and ψ. Let $\lambda \in \mathbb{R}^+$. The following are equivalent:*

1. *We have $\varphi\sigma_t^{\psi} = \lambda^t \varphi$, for all $t \in \mathbb{R}$.*
2. *We have $\psi\sigma_t^{\varphi} = \lambda^{-t}\psi$, for all $t \in \mathbb{R}$.*
3. *There exists a strictly positive operator Q that is affiliated with M such that $\sigma_t^{\varphi}(Q^{is}) = \lambda^{ist} Q^{is}$ for all $s, t \in \mathbb{R}$ and such that $\psi = \varphi_Q$.*
4. *There exists a strictly positive operator Q that is affiliated with M such that $\left(\frac{D\psi}{D\varphi}\right)_t = \lambda^{\frac{1}{2}it^2} Q^{it}$ for all $t \in \mathbb{R}$.*

A.5 Connes' spatial derivative

In this section we recall the construction of the *spatial derivative*. Therefore, fix a von Neumann algebra M and consider a normal, semi-finite, not necessarily faithful weight φ on M and a normal, semi-finite, faithful weight ϕ on M'. Let $(\mathcal{H}, \pi, \Lambda)$ be the GNS construction of φ. Let $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ be the GNS construction for ϕ.

Definition A.5.1. A vector $\xi \in \mathcal{H}$ is called *ϕ-bounded* if the map $\Lambda_\phi(x) \mapsto x\xi, x \in \mathfrak{n}_\phi$ extends to a bounded map $\mathcal{H}_\phi \to \mathcal{H}$. This map will be denoted by $R^{\phi}(\xi)$. Denote the set of ϕ-bounded vectors by $D(\mathcal{H}, \phi)$.

Let $\xi \in \mathcal{H}$ be ϕ-bounded. It is easy to check that

$$R^{\phi}(\xi)\pi_{\phi}(x) = xR^{\phi}(\xi), \qquad x \in M'.$$

It follows that if $\xi, \eta \in \mathcal{H}$ are ϕ-bounded, then the operator $R^{\phi}(\xi)^* R^{\phi}(\eta) : \mathcal{H} \to \mathcal{H}$ is in M. In particular $R^{\phi}(\xi)^* R^{\phi}(\xi)$ is positive. We can consider the following quadratic form

$$q(\xi) = \begin{cases} \varphi(R(\xi)^* R(\xi)) & \xi \in D(\mathcal{H}, \phi), \\ \infty & \text{otherwise.} \end{cases}$$

Now, we have the following theorem.

Theorem A.5.2. *q is a closed, densely defined, positive quadratic form with domain Dom(q) which equals the ϕ-bounded vectors for which $q(\xi) < \infty$.*

From the general theory of quadratic forms, there exists a unique positive, self-adjoint operator $\frac{d\varphi}{d\phi}$ such that the domain of $\left(\frac{d\varphi}{d\phi}\right)^{\frac{1}{2}}$ equals the domain of $\mathrm{Dom}(q)$ and such that:

$$q(\xi) = \langle \frac{d\varphi}{d\phi}^{\frac{1}{2}} \xi, \frac{d\varphi}{d\phi}^{\frac{1}{2}} \xi \rangle, \qquad \xi \in \mathrm{Dom}(q). \tag{A.2}$$

The operator $\frac{d\varphi}{d\phi}$ is called the *spatial derivative* of φ with respect to ϕ. Note that in particular (A.2) allows one to explicitly compute the spatial derivative. Probably the most important property we need is that under the assumption that φ is faithful, $\frac{d\varphi}{d\phi}$ implements both the modular automorphisms σ^{φ} and σ^{ϕ}, that is:

$$\sigma_t^{\varphi}(x) = \left(\frac{d\varphi}{d\phi}\right)^{it} x \left(\frac{d\varphi}{d\phi}\right)^{-it}, \qquad x \in M,$$
$$\sigma_t^{\phi}(y) = \left(\frac{d\varphi}{d\phi}\right)^{-it} y \left(\frac{d\varphi}{d\phi}\right)^{it}, \qquad y \in M'.$$

In particular, if ϕ is tracial, then $\frac{d\varphi}{d\phi}$ is affiliated with M and if φ is tracial, then $\frac{d\varphi}{d\phi}$ is affiliated with M'.

A.6 Approximation lemmas

The following lemmas are probably very well known. However, we did not found them explicitly in the literature. For completeness, we prove them here. Here M is a von Neumann algebra with normal, semi-finite, faithful weight φ.

The following lemma is a consequence of [74, Theorem II.2.6]. As a corollary, we see that the anti-pode of a von Neumann algebraic quantum group is σ-weak/σ-weak closed.

Lemma A.6.1. *Let M be a von Neumann algebra and let $S : Dom(S) \to M$ be a linear map with $Dom(S) \subseteq M$. S is σ-weak/σ-weak closed if and only if S is σ-strong-$*$/σ-strong-$*$ closed.*

Proof. The only if part is trivial. For $(x, y) \in M \times M$, we put $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. Let $p_1 : M \times M \to M : (x, y) \mapsto x$ and $p_2 : M \times M \to M : (x, y) \mapsto y$ denote the projections on the first and second coordinate. If $f \in (M \times M)^*$, then there are $f_1, f_2 \in M^*$ such that $f = f_1 p_1 + f_2 p_2$. From [74, Chapter II, Theorem 2.6], we see that the following are equivalent.

1. f is continuous with respect to the product topology of the σ-strong-$*$ topology on M.
2. f_1 and f_2 are continuous with respect to the σ-strong-$*$ topology on M.
3. f_1 and f_2 are continuous with respect to the σ-weak topology on M.
4. f is continuous with respect to the product topology of the σ-weak topology on M.

Therefore, a convex set in $M \times M$ is closed with respect to the product of the σ-strong-$*$ topology on M if and only if it is closed with respect to the product of the σ-weak topology on M, see [13, Chapter IV, Theorem 3.7]. Applying this to the graph of S yields the lemma. □

Next, we need the following variation on [78, Lemma 9].

Lemma A.6.2. *Let $\delta > 0$. There exists a net $(e_j)_{j \in J}$ in $\mathcal{T}_\varphi$ such that* (1) $\|\sigma_z(e_j)\| \leq e^{\delta \mathrm{Im}(z)^2}$, (2) $e_j \to 1$ *strongly and* (3) $\sigma_{i/2}(e_j) \to 1$ *σ-weakly.*

Proof. Let $(f_j)_{j \in J}$ and $(e_j)_{j \in J}$ be nets as in [78, Lemma 9]. This lemma proves already that $(e_j)_{j \in J}$ satisfies (1) and (2). Now, (3) follows, since for $\xi \in \mathcal{H}$,

$$
\begin{aligned}
\langle \sigma_{\frac{i}{2}}(e_j)\xi, \xi \rangle =& \omega_{\xi,\xi}\left(\sqrt{\frac{\delta}{\pi}} \int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2} \sigma_t(f_j) dt \right) \\
=& \left(\sqrt{\frac{\delta}{\pi}} \int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2} (\omega_{\xi,\xi} \circ \sigma_t) dt \right)(f_j) \\
\to& \left(\sqrt{\frac{\delta}{\pi}} \int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2} (\omega_{\xi,\xi} \circ \sigma_t) dt \right)(1) \\
=& \sqrt{\frac{\delta}{\pi}} \int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2} dt \langle \xi, \xi \rangle \\
=& \langle \xi, \xi \rangle,
\end{aligned}
$$

where the last equality is obtained by means of the residue formula for meromorphic functions. So $\sigma_{\frac{i}{2}}(e_k)$ is bounded and converges weakly, hence σ-weakly to 1. □

Lemma A.6.3. *$\mathcal{T}_\varphi^2$ is a σ-strong-$*$/norm core for Λ. Furthermore, for every $x \in \mathfrak{n}_\varphi$, there is a bounded net $(a_j)_{j\in J}$ in $\mathcal{T}_\varphi^2$ such that $a_j \to x$ σ-weakly and $\Lambda(a_j) \to \Lambda(x)$ weakly.*

Proof. For the first statement, it is enough to prove that $\mathcal{T}_\varphi^2$ is a σ-weak/weak core for Λ, since the σ-weak/weak continuous functionals on the graph of Λ equal the σ-strong-$*$/norm continuous functionals, c.f. the proof of Lemma A.6.1. Therefore, it is enough to prove the second statement.

First of all, since φ is semi-finite and using Kaplanski's density theorem, let $(b_j)_{j\in J}$ be bounded net in $\mathfrak{n}_\varphi^*$ that converges strongly (hence σ-weakly) to 1. Then $(b_j x)_{j\in J}$ is a bounded net in $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ such that $b_j x \to x$ σ-weakly and $\Lambda(b_j x) \to \Lambda(x)$ weakly. From this observation we see that we have to prove the second statement only for the case $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

So assume that $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. Put

$$x_n = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(nt)^2} \sigma_t(x) dt,$$

where the integral is taken in the σ-strong-$*$ sense. By standard techniques (c.f. the proof of [78, Lemma 9]), $x_n \in \mathcal{T}_\varphi$ and x_n converges σ-weakly to x. Moreover, using the the fact that Λ is σ-strong-$*$/norm closed, we obtain

$$\Lambda(x_n) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(nt)^2} \nabla^{it} \Lambda(x) dt \to \Lambda(x) \qquad \text{weakly},$$

where the integral is a Bochner integral, c.f. [75, Chapter VI, Lemma 2.4]. Clearly, x_n is a bounded sequence. From this observation, we see that it is enough to prove the second statement for the case $x \in \mathcal{T}_\varphi$. Now, let $x \in \mathcal{T}_\varphi$ and let $(e_j)_{j\in J}$ be a bounded net in $\mathcal{T}_\varphi$ such that $e_j \to 1$ σ-weakly, c.f. Lemma A.6.2. Then, $e_j x \in \mathcal{T}_\varphi^2$ and $e_j x \to x$ σ-weakly and $\Lambda(e_j x) = e_j \Lambda(x) \to \Lambda(x)$ weakly. This net is clearly bounded. $\square$

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Nederlandse samenvatting

In dit proefschrift worden drie verschillende onderwerpen binnen de wiskunde bestudeerd: *kwantumgroepen*, *Fourier theorie* en *niet-commutatieve L^p-ruimten*. We leggen verbanden tussen deze begrippen met behulp van technieken afkomstig uit vakgebieden als operatoralgebra's en functionaalanalyse. In dit hoofdstuk staat een korte inleiding op deze begrippen en een schets van de resultaten die in dit proefschrift te vinden zijn.

Fourier theorie is vernoemd naar de Franse wis- en natuurkundige Jean-Baptiste Joseph Fourier (21 maart 1768 – 16 mei 1830). Dit domein binnen de wiskunde houdt zich bezig met de decompositie van functies in elementaire 'golffuncties'. Het meest bekende voorbeeld is het volgende. Een zekere klasse van continue functies $f \in L^1(\mathbb{R})$ laat een decompositie toe van de vorm

$$f(x) = \int_{\mathbb{R}} c(y) e^{ixy} dy,$$

waarbij $c(y) = \int_{\mathbb{R}} f(z) e^{-izy} dz$. Deze formule zegt feitelijk dat f is opgebouwd uit een continuüm van golffuncties. In dit geval zijn de golffuncties geparametriseerd door $y \in \mathbb{R}$ en worden gegeven door $x \mapsto e^{ixy}$.

Dergelijke decomposities komen op allerlei plekken in de wis- en natuurkunde terug. Niet in de minste plaats vormt het de basis van de kwantummechanica, waarin de beweging van kleine deeltjes beschreven wordt aan de hand van oplossingen van de Schrödinger vergelijking, ofwel de elementaire golffuncties.

De keuze voor de golffuncties $x \mapsto e^{ixy}$ als basis van een decompositie is enigszins beperkend. Dankzij een stelling uit de abstracte harmonische analyse weten we dat we de functies $x \mapsto e^{ixy}$ met $y \in \mathbb{R}$ kunnen vervangen door matrix elementen van representaties van groepen. In dat geval dient ook $\mathbb{R}$ vervangen te worden door de betreffende groep. De stelling waarnaar hier verwezen wordt, staat bekend als de *Plancherel stelling* en speelt op veel plekken in dit proefschrift een rol.

Het tweede centrale object in dit proefschrift wordt gevormd door *kwantumgroepen*. Kwantumgroepen zijn in de tweede helft van de twintigste eeuw ontstaan. De idee achter kwantumgroepen is het volgende. Men beschouwt de continue functies die verdwijnen in oneindig op een lokaal compacte groep

G, noem dit $C_0(G)$. Alle topologische gegevens van G zijn terug te vinden in $C_0(G)$, zoals volgt uit de Gelfand-Naimark stelling. Met wat extra structuur is ook de groepsvermenigvuldiging, de eenheid en de inversie van G in $C_0(G)$ te bevatten. Het volstaat dus $C_0(G)$ te bestuderen in plaats van G zelf. Kwantumgroepen zijn (in veel gevallen) *deformaties* van $C_0(G)$. Het zijn C*-algebras met extra structuur gelijkend op die van $C_0(G)$.

Er is ook een meetbare variant van kwantumgroepen. In dat geval deformeert men meetbare functies en eindigt men met een von Neumann algebra. De continue en meetbare benadering hebben belangrijke relaties en versterken elkaar in veel gevallen.

Het opmerkelijke is nu dat een enorme hoeveelheid gereedschap uit de abstracte harmonische analyse een passende interpretatie blijkt te hebben voor kwantumgroepen. Dit is één van de redenen om kwantumgroepen te bestuderen en in dit proefschrift wellicht de belangrijkste reden. In het bijzonder heeft de Plancherel stelling (of Plancherel decompositie) een interpretatie in de kwantumwereld.

Tot slot speelt de theorie van *niet-commutatieve integratie* een belangrijke rol. Hier bestudeert men von Neumann algebras in plaats van maatruimten. Dit zijn zekere deelalgebras van de begrensde operatoren op een Hilbertruimte, denk bijvoorbeeld aan een matrix algebra. In het bijzonder geldt voor twee operatoren (of matrices) A en B in het algemeen niet dat $AB = BA$. Vandaar dat de theorie *niet-commutatief* wordt genoemd. Integralen worden nu vervangen door gewichten; bepaalde onbegrensde $\mathbb{C}$-waardige lineaire afbeeldingen (functionalen) op een von Neumann algebra. Denk bijvoorbeeld aan het spoor van matrices. Echter, voor een gewicht φ en operatoren A en B geldt in het algemeen niet dat

$$\varphi(AB) = \varphi(BA).$$

Wanneer deze vergelijking wel stand houdt voor voldoende A en B, dan noemt met φ een spoor (Engels: trace).

In het bijzonder vertaalt de Haarmaat op een groep zich in een *Haargewicht* op een kwantumgroep. Haargewichten op de kwantumgroepen die in dit proefschrift voorkomen, zijn typisch niet een spoor. We zijn dan ook voornamelijk geïnteresseerd in niet-commutatieve integratie met gewichten die geen spoor zijn.

In dit proefschrift worden verbanden gelegd tussen Fourier theorie, kwantumgroepen en niet-commutatieve integratie. Soms wordt één enkel van deze onderwerpen bekeken, soms twee en soms zelfs alle drie tegelijk. We lichten kort de inhoud van ieder hoofdstuk toe.

Hoofdstuk 2. Voor een willkeurig gewicht φ (dus niet noodzakelijk een spoor) en operatoren A en B kan men bewijzen dat er een 'getwiste gelijkheid' stand houdt:

$$\varphi(AB) = \varphi(B\sigma_{-i}(A)).$$

Hierbij is σ de zogenaamde *modulaire automorfisme groep* van een gewicht. Het bewijs van deze getwiste gelijkheid berust op een diep resultaat dat bekend staat als Tomita-Takesaki theorie.

Hoofdstuk 2 laat zien hoe deze modulaire automorfisme groep kan worden uitgedrukt in termen van de Plancherel decompositie van een kwantumgroep. We geven tevens enkele toepassingen. In het bijzonder bepalen we het Haargewicht van een specifieke kwantumgroep.

Hoofdstukken 3 and 4. We geven het noodzakelijke kader voor een studie van *sferische Fourier transformaties* en een *sferische Plancherel stelling* voor kwantumgroepen. Voor groepen staat het laatste ook wel bekend als de *Plancherel-Godement stelling*. De stelling geeft een decompositie voor functies op een groep die invariant zijn onder de actie van een ondergroep. Dergelijke stellingen waren al bekend in de gevallen aangegeven met een $\times$ in het volgende diagram.

	compact	niet compact
groepen	$\times$	$\times$
kwantumgroepen	$\times$	

Het juiste kader in het geval 'kwantumgroep + niet compact' vergt een nieuwe benadering. Deze is gegeven in Hoofdstuk 3. Hoofdstuk 4 geeft hier een voorbeeld van.

Hoofdstuk 5. We bestuderen het volgende opmerkelijke resultaat. Als $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ voor zekere $1 \leq p \leq 2$, dan is

$$\hat{f}(y) = \int_{\mathbb{R}} f(z) e^{-izy} dz,$$

bevat in $L^q(\mathbb{R})$, waarbij $\frac{1}{p} + \frac{1}{q} = 1$. Sterker nog $\|\hat{f}\|_q \leq \|f\|_p$. De afbeelding $f \mapsto \hat{f}$ heet ook wel de *L^p-Fourier transformatie* en de ongelijkheid $\|\hat{f}\|_q \leq \|f\|_p$ staat bekend als de *Hausdorff-Young ongelijkheid*.

Hoofdstuk 5 gaat over een vergaande generalisatie van dit resultaat. We vinden de analoge transformatie en ongelijkheid voor kwantumgroepen. Tevens laten we zien dat de theorie van niet-commutatieve integratie van *essentieel* belang is voor het begrijpen van zulke transformaties.

Hoofdstuk 6. Het laatste hoofdstuk is gewijd aan een studie van Schauder bases in niet-commutatieve L^p-ruimten.

Curriculum Vitae

Martijn Caspers was born on July 3, 1985 in Oploo, the Netherlands. He started studying mathematics in 2003 at the Katholieke Universiteit Nijmegen, nowadays known as Radboud Universiteit Nijmegen. Under the supervision of Klaas Landsman he wrote his Master's thesis entitled 'Gelfand spectra of C^*-algebras in topos theory' for which he received his Master's degree in 2008 (cum laude). His thesis was awarded by the research cluster for Geometry and Quantum Theory (GQT) for being the best thesis in that year within the the area of the cluster. From September 2008 Martijn was funded by GQT in a Ph.D. project under supervision of Erik Koelink at the Radboud Universiteit Nijmegen. During his Ph.D. he was given the opportunity to stay for six months at the University of New South Wales in Sydney, Australia hosted by Fedor Sukochev.